On Contra-Precontinuous Functions

SAEID JAFARI AND TAKASHI NOIRI

Department of Mathematics and Physics, Roskilde University, Postbox 260, 4000 Roskilde, Denmark
Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501 Japan

1e-mail: sjafari@ruc.dk and 2e-mail: noiri@as.yatsushiro-nct.ac.jp

Abstract. The notion of contra-continuity was introduced and investigated by Dontchev [3]. In this paper, we introduce and investigate a new generalization of contra-continuity called contra-precontinuity.

1. Introduction

Dontchev [3] introduced the notions of contra-continuity and strong S-closedness in topological spaces. He defined a function \( f : X \to Y \) to be contra-continuous if the preimage of every open set of \( Y \) is closed in \( X \). In [3], he obtained very interesting and important results concerning contra-continuity, compactness, S-closedness and strong S-closedness. Recently a new weaker form of this class of functions called contra-semicontinuous functions is introduced and investigated by Dontchev and Noiri [5]. They also introduced the notion of \( RC \)-continuity [5] between topological spaces which is weaker than contra-continuity and stronger than \( B \)-continuity [35].Quite recently, the present authors [12] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of \( RC \)-continuous functions and contra-continuous functions.

The aim of this paper is to introduce and investigate a new class of functions called contra-precontinuous functions which is weaker than contra-continuous functions. In Section 3, we obtain several basic properties of contra-precontinuous functions. In Section 4, we introduce contra-preclosed graphs and investigate relations between contra-precontinuity and contra-preclosed graphs. In Section 5, we obtain some properties of strongly S-closed spaces and compact spaces. Decompositions of \( RC \)-continuity and perfect continuity are also obtained. In the last section, we deal with strong forms of connectedness.
2. Preliminaries

In what follows, spaces $X$ and $Y$ are always topological spaces. $\text{Cl}(A)$ and $\text{Int}(A)$ designate the closure and interior of $A$ which is a subset of $X$. A subset $A$ is said to be regular open (resp. regular closed) if $A = \text{Int} (\text{Cl}(A))$ (resp. $A = \text{Cl} (\text{Int}(A))$).

**Definition 2.1.** A subset $A$ of a space $X$ is called

(i) preopen [16] if $A \subseteq \text{Int} (\text{Cl}(A))$,

(ii) semi-open [15] if $A \subseteq \text{Cl} (\text{Int}(A))$,

(iii) $\alpha$-open [22] if $A \subseteq \text{Int} (\text{Cl} (\text{Int}(A)))$,

(iv) $\beta$-open [1] if $A \subseteq \text{Cl} (\text{Int} (\text{Cl}(A)))$.

The complement of a preopen (resp. semi-open, $\alpha$-open, $\beta$-open) set is said to be preclosed (resp. semi-closed, $\alpha$-closed, $\beta$-closed). The collection of all closed (resp. preopen, semi-open, $\alpha$-open and $\beta$-open) subsets of $X$ will be denoted by $\text{C}(X)$ (resp. $\text{PO}(X)$, $\text{SO}(X)$, $\alpha (X)$ and $\beta (X)$). We set $C(X, x) = \{V \in C(X) \mid x \in V\}$ for $x \in X$. We define similarly $\text{PO}(X, x)$, $\text{SO}(X, x)$, $\alpha(X, x)$ and $\beta(X, x)$.

**Definition 2.2.** A function $f : X \to Y$ is called perfectly continuous [23] (resp. RC-continuous [5]) if for each open set $V$ of $Y$, $f^{-1}(V)$ is clopen (resp. regular-closed) in $X$.

**Definition 2.3.** A function $f : X \to Y$ is called precontinuous [16] (resp. semi-continuous [15], $\beta$-continuous [1]) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X, x)$ (resp. $U \in \text{SO}(X, x)$, $U \in \beta(X, x)$) such that $f(U) \subseteq V$.

**Definition 2.4.** A function $f : X \to Y$ is called almost precontinuous [20] if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists $U \in \text{PO}(X, x)$ such that $f(U) \subseteq \text{Int}(\text{Cl}(V))$.

**Definition 2.5.** A function $f : X \to Y$ is called contra-precontinuous (resp. contra-continuous [3], contra-semicontinuous [5], contra-$\alpha$-continuous [11], contra-$\beta$-continuous [3]) if $f^{-1}(V)$ is preclosed (resp. closed, semi-closed, $\alpha$-closed, $\beta$-closed) in $X$ for each open set $V$ of $Y$. 
For the functions defined above, we have the following diagram:

\[
\text{perfectly continuous } \iff \text{ RC-continuous } \iff \text{ contra-continuous } \\
\downarrow \\
\text{contra-}\alpha\text{-continuous } \iff \text{ contra-precontinuous } \\
\downarrow \\
\text{contra-semicontinuous } \iff \text{ contra-}\beta\text{-continuous }
\]

**Remark 2.1.** It should be noticed that contra-precontinuity and precontinuity are independent notions as shown by the following examples due to Dontchev [2].

**Example 2.1.** A continuous function need not be contra-precontinuous. The identity function on the real line with the usual topology is an example of a continuous function which is not contra-precontinuous.

**Example 2.2.** A contra-precontinuous function need not be precontinuous. Let \( X = \{a, b\} \) be the Sierpinski space by setting \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, X\} \). The identity function \( f : (X, \tau) \to (X, \sigma) \) is contra-precontinuous. But it is neither precontinuous nor semi-continuous.

### 3. Some properties

**Definition 3.1.** Let \( A \) be a subset of a space \((X, \tau)\).

1. The set \( \bigcap \{U \in \tau \mid A \subseteq U\} \) is called the kernel of \( A \) [19] and is denoted by \( \ker(A) \).
2. The set \( \bigcap \{F \in X \mid A \subseteq F, F : \text{preclosed}\} \) is called the preclosure of \( A \) [7] and is denoted by \( p\text{Cl}(A) \).

**Lemma 3.1.** The following properties hold for subsets \( A, B \) of a space \( X \):

1. \( x \in \ker(A) \) if and only if \( A \cap F \neq \emptyset \) for any \( F \in C(X, x) \).
2. \( A \subseteq \ker(A) \) and \( A = \ker(A) \) if \( A \) is open in \( X \).
3. If \( A \subseteq B \), then \( \ker(A) \subseteq \ker(B) \).
**Theorem 3.1.** The following are equivalent for a function $f : X \to Y$:

1. $f$ is contra-precontinuous;
2. for every closed subset $F$ of $Y$, $f^{-1}(F) \in PO(X)$;
3. for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in PO(X, x)$ such that $f(U) \subset F$;
4. $f(pCl(A)) \subset \ker(f(A))$ for every subset $A$ of $X$;
5. $pCl(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset of $B$ of $Y$.

**Proof.** The implications (1) $\iff$ (2) and (2) $\implies$ (3) are obvious.

(3) $\implies$ (4): Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in PO(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \bigcup \{U_x \mid x \in f^{-1}(F)\} \in PO(X)$.

(2) $\implies$ (4): Let $A$ be any subset of $X$. Suppose that $y \notin \ker(f(A))$. Then by Lemma 3.1 there exists $F \in C(X, Y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $pCl(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(pCl(A)) \cap F = \emptyset$ and $y \notin f(pCl(A))$. This implies that $f(Cl(A)) \subset \ker(f(A))$.

(4) $\implies$ (5): Let $B$ be any subset of $Y$. By (4) and Lemma 3.1, we have $f(pCl(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and $pCl(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(5) $\implies$ (1): Let $V$ be any open set of $Y$. Then, by Lemma 3.1 we have $pCl(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $pCl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is preclosed in $X$.

**Theorem 3.2.** The following are equivalent for a function $f : X \to Y$:

(i) $f$ is contra-$\alpha$-continuous;
(ii) $f$ is contra-precontinuous and contra-semicontinuous.

**Proof.** This follows from the fact that $A \in \alpha(X)$ if and only if $A \in PO(X) \cap SO(X)$ [28, Lemma 1].

**Theorem 3.3.** If a function $f : X \to Y$ is contra-precontinuous and $Y$ is regular, then $f$ is precontinuous.
Proof. Let $x$ be an arbitrary point of $X$ and $V$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $\text{Cl}(W) \subset V$. Since $f$ is contra-precontinuous, so by Theorem 3.1 there exists $U \in \text{PO}(X, x)$ such that $f(U) \subset \text{Cl}(W)$. Then $f(U) \subset \text{Cl}(W) \subset V$. Hence, $f$ is precontinuous.

Remark 3.1. By Example 2.1, a precontinuous functions $f : X \to Y$ is not always contra-precontinuous even if $Y$ is regular.

Recall that a function $f : X \to Y$ is called $M$-preopen [17] if the image of each preopen set is preopen.

**Theorem 3.4.** If $f : X \to Y$ is an $M$-preopen contra-precontinuous function, then $f$ is almost precontinuous.

**Proof.** Let $x$ be any arbitrary point of $X$ and $V$ be an open neighborhood $f(x)$. Since $f$ is contra-precontinuous, then by Theorem 3.1 (3), there exists $U \in \text{PO}(X, x)$ such that $f(U) \subset \text{Cl}(V)$. Since $f$ is $M$-preopen, $f(U)$ is preopen in $Y$. Therefore $f(U) \subset \text{Int}(\text{Cl}(f(U))) \subset \text{Int}(\text{Cl}(V))$. This shows that $f$ is almost precontinuous.

**Definition 3.2.** A function $f : X \to Y$ is said to be almost weakly continuous [13] if $f^{-1}(V) \subset \text{Int}(f^{-1}(\text{Cl}(V)))$ for every open set $V$ of $Y$.

It is shown in [25, Theorem 3.1] that a function $f : X \to Y$ is almost weakly continuous if and only if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists $U \in \text{PO}(X, x)$ such that $f(U) \subset \text{Cl}(V)$.

**Remark 3.2.** The following implications are obvious:

precontinuity $\Rightarrow$ almost precontinuity $\Rightarrow$ almost weak continuity,

where the converses are false as shown in Examples 2.1 and 2.2 [10].

As shown in Example 2.2, a contra-precontinuous function need not be precontinuous. However, every contra-precontinuous function is necessarily almost weakly continuous.

**Theorem 3.5.** If a function $f : X \to Y$ is contra-precontinuous, then $f$ is almost weakly continuous.
Proof. Let $V$ be any open set of $Y$. Since $\text{Cl}(V)$ is closed in $Y$, $f^{-1}(\text{Cl}(V))$ is preopen in $X$ and we have $f^{-1}(V) \subset f^{-1}(\text{Cl}(V)) \subset \text{Int}(f^{-1}(\text{Cl}(V)))$. This shows that $f$ is almost weakly continuous.

The prefrontier [26] $p\text{Fr}(A)$ of $A$, where $A \subset X$, is defined by $p\text{Fr}(A) = p\text{Cl}(A) \cap p\text{Cl}(X - A)$.

**Theorem 3.6.** The set of all points of $x$ of $X$ at which $f : X \to Y$ is not contra-precontinuous is identical with the union of the prefrontier of the inverse images of closed sets of $Y$ containing $f(x)$.

Proof. “Necessity”. Suppose that $f$ is not contra-precontinuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in PO(X, x)$. This implies that $U \cap f^{-1}(Y - F) \neq \emptyset$. Therefore, we have $x \in p\text{Cl}(f^{-1}(Y - F)) = p\text{Cl}(X - f^{-1}(F))$. However, since $x \in f^{-1}(F), x \in p\text{Cl}(f^{-1}(F))$. Therefore, we obtain $x \in p\text{Fr}(f^{-1}(F))$.

“Sufficiency”. Suppose that $x \in p\text{Fr}(f^{-1}(F))$ for some $F \in C(Y, f(x))$. Now, we assume that $f$ is contra-precontinuous at $x$. Then there exists $U \in PO(X, x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in p\text{Int}(f^{-1}(F)) \subset X - p\text{Fr}(f^{-1}(F))$. This is a contradiction. This means that $f$ is not contra-precontinuous.

Recall that a family $E$ of subsets of a space $(X, \tau)$ is called a network for a topology $\tau$ on $(X, \tau)$ if every set in $\tau$ is the union of some subfamily of $E$.

**Definition 3.3.** A space $(X, \tau)$ is said to be

1. extremally disconnected [33] if the closure of every open set of $X$ is open in $X$,
2. locally indiscrete [21] if every open set of $X$ is closed in $X$,
3. submaximal [29] if every dense set of $X$ is open in $X$, equivalently if every preopen set is open,
4. strongly irresolvable [8] if no nonempty open set is resolvable, equivalently if every preopen subset is $\alpha$-open,
5. mildly Hausdorff [6] if the $\delta$-closed sets form a network for its topology $\tau$, where a $\delta$-closed set is the intersection of regular closed sets,
6. strongly S-closed [3] if every closed cover of $X$ has a finite subcover,
7. door space [4] if every subset of $X$ is either open or closed.
Remark 3.3. It should be noted that every door space $X$ is submaximal \cite[Theorem 2.7]{4} and every mildly Hausdorff strongly $S$-closed space is locally indiscrete \cite[6]{6}.

The following results follow immediately from Definition 3.3 and Remark 3.3:

**Theorem 3.7.** If a function $f : X \to Y$ is continuous and $X$ is locally indiscrete, then $f$ is contra-continuous.

**Corollary 3.1.** If a function $f : X \to Y$ is continuous and $X$ is mildly Hausdorff strongly $S$-closed, then $f$ is contra-continuous.

**Theorem 3.8.** Let $f : X \to Y$ be a contra-precontinuous function.

1. If $X$ is submaximal, then $f$ is contra-continuous,
2. If $X$ is strongly irresolvable, then $f$ is contra-$\alpha$-continuous.

**Corollary 3.2.** If a function $f : X \to Y$ is contra-precontinuous and $X$ is a door space, then $f$ is contra-continuous.

**Lemma 3.2.** For a subset $A$ of a space $X$, the following are equivalent:

1. $A$ is regular closed;
2. $A$ is preclosed and semi-open;
3. $A$ is $\alpha$-closed and $\beta$-open.

**Proof.** (1) $\Rightarrow$ (2): Let $A$ be regular closed. Then $A = \text{Cl}(\text{Int}(A))$ and $A$ is preclosed and semi-open.

(2) $\Rightarrow$ (3): Let $A$ be preclosed and semi-open. Then $\text{Cl}(\text{Int}(A)) \subseteq A$ and $A \subseteq \text{Cl}(\text{Int}(A))$. Therefore, we have $\text{Cl}(\text{Int}(A)) = \text{Cl}(A)$ and hence $\text{Cl}(\text{Int}(\text{Cl}(A))) = \text{Cl}(\text{Int}(\text{Cl}(A))) = \text{Cl}(\text{Int}(A)) \subseteq A$. This shows that $A$ is $\alpha$-closed. Since $\text{SO}(X) \subseteq \beta O(X)$, it is obvious that $A$ is $\beta$-open.

(3) $\Rightarrow$ (1): Let $A$ be $\alpha$-closed and $\beta$-open. Then $A = \text{Cl}(\text{Int}(\text{Cl}(A)))$ and hence $\text{Cl}(\text{Int}(A)) = \text{Cl}(\text{Int}(\text{Cl}(\text{Int}(A)))) = \text{Cl}(\text{Int}(\text{Cl}(A))) = A$. Therefore, $A$ is regular closed.

As a consequence of the above lemma, we have the following result:

**Theorem 3.9.** The following statements are equivalent for a function $f : X \to Y$:

1. $f$ is $RC$-continuous;
2. $f$ is contra-precontinuous and semi-continuous;
3. $f$ is contra-$\alpha$-continuous and $\beta$-continuous.
4. Contra-preclosed graphs

We begin with the following notion:

Definition 4.1. The graph \( G(f) \) of a function \( f : X \to Y \) is said to be contra-preclosed if for each \( (x, y) \in (X \times Y) - G(f) \), there exist \( U \in PO(X, x) \) and \( V \in C(Y, y) \) such that \( (U \times V) \cap G(f) = \emptyset \).

Lemma 4.1. The graph \( G(f) \) of \( f : X \to Y \) is contra-preclosed in \( X \times Y \) if and only if for each \( (x, y) \in (X \times Y) - G(f) \), there exist \( U \in PO(X, x) \) and \( V \in C(Y, y) \) such that \( f(U) \cap V = \emptyset \).

Theorem 4.1. If \( f : X \to Y \) is contra-precontinuous and \( Y \) is Urysohn, then \( G(f) \) is contra-preclosed in \( X \times Y \).

Proof. Let \( (x, y) \in (X \times Y) - G(f) \). Then \( y \neq f(x) \) and there exist open sets \( V, W \) such that \( f(x) \in V, y \in W \) and \( Cl(V) \cap Cl(W) = \emptyset \). Since \( f \) is contra-precontinuous, there exists \( U \in PO(X, x) \) such that \( f(U) \subseteq Cl(V) \). Therefore, we obtain \( f(U) \cap Cl(W) = \emptyset \). This shows that \( G(f) \) is contra-preclosed.

Definition 4.2. A space \( X \) is said to be

1. strongly compact [17] if every preopen cover of \( X \) has a finite subcover,
2. \( S \)-closed [34] if every semi-open cover \( \{ V_\alpha \mid \alpha \in \mathbb{N} \} \) of \( X \), there exists a finite subset \( \mathbb{N}_0 \) of \( \mathbb{N} \) such that \( X = \bigcup \{ Cl(V_\alpha) \mid \alpha \in \mathbb{N}_0 \} \), equivalently if every regular closed cover of \( X \) has a finite subcover,
3. mildly compact [32] if every clopen cover of \( X \) has a finite subcover.

Definition 4.3. A subset \( S \) of a space \( X \) is said to be

1. strongly compact relative to \( X \) [17] if every cover of \( S \) by preopen sets of \( X \) has a finite subcover,
2. strongly \( S \)-closed [3] if the subspace \( S \) is strongly \( S \)-closed.

Theorem 4.2. Let \( X \) be submaximal. If \( f : X \to Y \) has a contra-preclosed graph, then the inverse image of a strongly \( S \)-closed set \( K \) of \( Y \) is closed in \( X \).

Proof. Assume that \( K \) is a strongly \( S \)-closed set of \( Y \) and \( x \notin f^{-1}(K) \). For each \( k \in K, (x, k) \notin G(f) \). By Lemma 4.1, there exists \( U_k \in PO(X, x) \) and \( V_k \in C(Y, k) \) such that \( f(U_k) \cap V_k = \emptyset \). Since \( \{ K \cap V_k \mid k \in K \} \) is a closed cover of the subspace \( K \), there exists a finite subset \( K_1 \subseteq K \) such that
On Contra-Precontinuous Functions

123

\( K \subset \bigcup \{ V_k \mid k \in K \} \). Set \( U = \bigcap \{ U_k \mid k \in K \} \), then \( U \) is open since \( X \) is submaximal. Therefore \( f(U) \cap K = \emptyset \) and \( U \cap f^{-1}(K) = \emptyset \). This shows that \( f^{-1}(K) \) is closed in \( X \).

A space \( X \) is said to be weakly Hausdorff [31] if each point of \( X \) is an intersection of regular closed sets of \( X \).

**Corollary 4.1.** Let \( X \) be submaximal and \( Y \) be strongly \( S \)-closed weakly Hausdorff. The following properties are equivalent for a function \( f : X \to Y \):

1. \( f \) is contra-precontinuous;
2. \( G(f) \) is contra-preclosed;
3. \( f^{-1}(K) \) is closed in \( X \) for every strong \( S \)-closed set \( K \) of \( Y \);
4. \( f \) is contra-continuous.

**Proof.**

1. \( (1) \Rightarrow (2) \): It is shown in [9, Theorem 3.7] that every \( S \)-closed weakly Hausdorff space is extremally disconnected. Since a strongly \( S \)-closed space is \( S \)-closed, \( Y \) is extremally disconnected and hence every regular closed set of \( Y \) is clopen. This shows that \( Y \) is Urysohn. By Theorem 4.1, \( G(f) \) is contra-preclosed.

2. \( (2) \Rightarrow (3) \): This is a result of Theorem 4.2.

3. \( (3) \Rightarrow (4) \): First, we show that an open set of \( Y \) is strongly \( S \)-closed. Let \( V \) be an open set of \( Y \) and \( \{ H_\alpha \mid \alpha \in \mathbb{V} \} \) be a cover of \( V \) by closed sets \( H_\alpha \) of the subspace \( V \). For each \( \alpha \in \mathbb{V} \), there exists a closed set \( K_\alpha \) of \( X \) such that \( H_\alpha = K_\alpha \cap V \). Then, the family \( \{ K_\alpha \mid \alpha \in \mathbb{V} \} \cup (Y - V) \) is a closed cover of \( Y \). Since \( Y \) is strongly \( S \)-closed, there exists a finite subset \( \mathbb{V}_+ \subset \mathbb{V} \) such that \( Y = \bigcup \{ K_\alpha \mid \alpha \in \mathbb{V}_+ \} \cup (Y - V) \). Therefore we obtain \( V = (\bigcup \{ K_\alpha \mid \alpha \in \mathbb{V}_+ \}) \cap V = \bigcup \{ H_\alpha \mid \alpha \in \mathbb{V}_+ \} \). This shows that \( V \) is strongly \( S \)-closed. For any open set \( V \), by (3) \( f^{-1}(V) \) is closed in \( X \) and \( f \) is contra-continuous.

5. **Strong forms of compactness**

**Theorem 5.1.** If \( f : X \to Y \) is contra-precontinuous and \( K \) is strongly compact relative to \( X \), then \( f(K) \) is strongly \( S \)-closed in \( Y \).

**Proof.** Let \( \{ H_\alpha \mid \alpha \in \mathbb{V} \} \) be any cover of \( f(K) \) by closed sets of the subspace \( f(K) \). For each \( \alpha \in \mathbb{V} \), there exists a closed set \( K_\alpha \) of \( Y \) such that \( H_\alpha = K_\alpha \cap f(K) \). For each \( x \in K \), there exists \( \alpha(x) \in \mathbb{V} \) such that
f(x) ∈ K_{α(x)} and by Theorem 3.1 there exists U_x ∈ PO(X, x) such that f(U_x) ⊂ K_{α(x)}. Since the family \{U_x \mid x ∈ K\} is a preopen cover of K, there exists a finite subset \( K_0 \) of K such that \( K ⊂ \bigcup \{U_x \mid x ∈ K_0\}. \) Therefore, we obtain \( f(K) ⊂ \bigcup \{f(U_x) \mid x ∈ K_0\} \) which is a subset of \( \bigcup \{K_{α(x)} \mid α ∈ K_0\}. \) Thus, \( f(K) = \bigcup \{H_{α(x)} \mid x ∈ K_0\} \) and hence \( f(K) \) is strongly S-closed.

**Corollary 5.1.** If \( f : X → Y \) is contra-precontinuous surjection and X is strongly compact, then Y is strongly S-closed.

**Theorem 5.2.** A function \( f : X → Y \) is RC-continuous if and only if it is contra-precontinuous and semi-continuous.

**Proof.** “Necessity”. Every RC-continuous function is contra-continuous and hence contra-precontinuous. Since every regular closed set is semi-open, RC-continuous functions are semi-continuous.

“Sufficiency”. For any open set V of Y, \( f^{-1}(V) \) is preclosed and semi-open in X and hence we have \( Cl(\text{Int}(f^{-1}(V))) ⊂ f^{-1}(V) ⊂ Cl(\text{Int}(f^{-1}(V))). \) Therefore, we obtain \( Cl(\text{Int}(f^{-1}(V))) = f^{-1}(V) \) and hence \( f \) is RC-continuous.

**Remark 5.1.** It follows from Examples 2.1 and 2.2 that contra-precontinuity and semi-continuity are independent of each other. Therefore, by Theorem 5.2 we had a decomposition of RC-continuity.

**Theorem 5.3.** If \( f : X → Y \) is an RC-continuous surjection and X is S-closed, then Y is compact.

**Proof.** Let \( \{V_α \mid α ∈ \mathcal{V}\} \) be any open cover of Y. Then \( \{f^{-1}(V_α) \mid α ∈ \mathcal{V}\} \) is a regular closed cover of X and we have \( X = \bigcup \{f^{-1}(V_α) \mid α ∈ \mathcal{V}\} \) for some finite subset \( \mathcal{V}_0 \) of \( \mathcal{V}. \) Since \( f \) is surjective, \( Y = \bigcup \{V_α \mid α ∈ \mathcal{V}_0\} \) and Y is compact.

A function \( f : X → Y \) is said to be \( α \)-continuous \([18]\) if \( f^{-1}(V) ∈ α(X) \) for every open set V of Y. In [2, Theorem 2.9], Dontchev obtained decompositions of perfect continuity. The following is also a decomposition of perfect continuity.

**Theorem 5.4.** A function \( f : X → Y \) is perfectly continuous if and only if it is contra-precontinuous and \( α \)-continuous.
Proof. “Necessity”. This is obvious.
“Sufficiency”. Let \( f \) be contra-precontinuous and \( \alpha \)-continuous. Let \( V \) be any open set of \( Y \). Then \( f^{-1}(V) \) is preclosed and \( \alpha \)-open in \( X \). Therefore, we have

\[
 \text{Int} ( \text{Cl}(\text{Int}(f^{-1}(V)))) \subseteq \text{Cl}( \text{Int}(f^{-1}(V))) \subseteq f^{-1}(V) \\
\subseteq \text{Int}( \text{Cl}(\text{Int}(f^{-1}(V)))) \subseteq \text{Cl}( \text{Int}(f^{-1}(V))).
\]

This implies that \( f^{-1}(V) \) is clopen in \( X \). Thus, \( f \) is perfectly continuous.

**Remark 5.2.** Contra-precontinuity and \( \alpha \)-continuity are independent of each other as shown by Examples 2.1 and 2.2.

**Corollary 5.2.** (Dontchev [3]). For a function \( f : X \to Y \) the following are equivalent:
1. \( f \) is perfectly continuous;
2. \( f \) is continuous and contra-continuous;
3. \( f \) is \( \alpha \)-continuous and contra-continuous.

**Theorem 5.5.** If \( f : X \to Y \) is a perfectly continuous surjection and \( X \) is mildly compact, then \( Y \) is compact.

Proof. Let \( f : X \to Y \) be a perfectly continuous surjection and \( X \) be mildly compact. Let \( \{V_\alpha \mid \alpha \in \mathcal{V}\} \) be any open cover of \( Y \). Then \( \{f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V}\} \) is a clopen cover of \( X \). Since \( X \) is mildly compact, there exists a finite subset \( \mathcal{V}_* \) of \( \mathcal{V} \) such that \( X = \bigcup \{f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V}_*\} \). Since \( f \) is surjective, \( Y = \bigcup \{V_\alpha \mid \alpha \in \mathcal{V}_*\} \) and \( Y \) is compact.

A space \( X \) is said to be almost compact [30] or quasi \( H \)-closed [27] if every open cover has a finite subfamily the closures of whose members cover \( X \). We have the following implications:

-strongly \( S \)-closed \( \Rightarrow \) \( S \)-closed \( \Rightarrow \) almost compact \( \Rightarrow \) mildly compact.

**Corollary 5.3.** (Dontchev [3]). The image of an almost compact space under contra-continuous nearly continuous (precontinuous) function is compact.

Proof. It is shown in [3, Theorem 2.9] that a function is contra-continuous and nearly continuous if and only if it is perfectly continuous. Then, the proof follows from Theorem 5.5.
6. Strong forms of connectedness

Definition 6.1. A space $X$ is said to be

1. hyperconnected \[33\] if $\text{Cl}(V) = X$ for every nonempty open set $V$ of $X$,
2. $\theta$-irreducible \[14\] if every two nonempty regular closed sets intersect,
3. preconnected \[24\] if $X$ cannot be expressed as the union of two nonempty preopen sets.

Theorem 6.1. Let $X$ be preconnected and $Y$ be $T_1$. If $f : X \to Y$ is contra-precontinuous, then $f$ is constant.

Proof. Since $Y$ is $T_1$-space, $U = \{f^{-1}(y) \mid y \in Y\}$ is a disjoint preopen partition of $X$. If $\left|U\right| \geq 2$, then $X$ is the union of two nonempty preopen sets. Since $X$ is preconnected, $\left|U\right| = 1$. Therefore, $f$ is constant.

A function $f : X \to Y$ is said to be preclosed \[7\] if the image $f(A)$ is preclosed in $Y$ for every closed set $A$ of $X$.

Theorem 6.2. Let $f : X \to Y$ be a contra-precontinuous and preclosed surjection. If $X$ is submaximal, then $Y$ is locally indiscrete.

Proof. Let $V$ be any open set of $Y$. Since $f$ is contra-continuous and $X$ is submaximal, $f^{-1}(V)$ is closed in $X$ and hence $V$ is preclosed in $Y$. Therefore, $\text{Cl}(V) = \text{Cl}(\text{Int}(V)) \subset V$ and $V$ is closed in $Y$. This shows that $Y$ is locally indiscrete.

Theorem 6.3. If $f : X \to Y$ is a contra-precontinuous semi-continuous surjection and $X$ is $\theta$-irreducible, then $Y$ is hyperconnected.

Proof. Suppose that $Y$ is not hyperconnected. Then, there exists two disjoint nonempty open sets $V$, $W$ and $Y$. By Theorem 5.2, $f$ is an $RC$-continuous surjection and $f^{-1}(V)$, $f^{-1}(W)$ are disjoint nonempty regular closed sets of $X$. Therefore, $X$ is not $\theta$-irreducible.

Theorem 6.4. If $f : X \to Y$ is a contra-precontinuous $\alpha$-continuous surjection and $X$ is connected, then $Y$ has an indiscrete topology.

Proof. Suppose that there exists a proper open set $V$ of $Y$. By Theorem 5.4, $f$ is a perfectly continuous surjection and $f^{-1}(V)$ is a proper clopen set of $X$. This shows that $X$ is not connected. Therefore, $Y$ has an indiscrete topology.

Acknowledgement. The authors are very grateful to the referee for his careful work.
On Contra-Precontinuous Functions

References


Keywords and phrases: contra-precontinuous, preopen set, mildly compact.

1991 AMS Subject Classification: 54C08