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On Contra-Precontinuous Functions

[1]SAEID JAFARI AND [2]TAKASHI NOIRI

[1]Department of Mathematics and Physics, Roskilde University, Postbox 260, 4000 Roskilde, Denmark
[2]Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501 Japan
[1]e-mail: sjafari@ruc.dk and [2]e-mail: noiri@as.yatsushiro-nct.ac.jp

Abstract. The notion of contra-continuity was introduced and investigated by Dontchev [3]. In this paper, we introduce and investigate a new generalization of contra-continuity called contra-precontinuity.

1. Introduction

Dontchev [3] introduced the notions of contra-continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ to be contra-continuous if the preimage of every open set of Y is closed in X. In [3], he obtained very interesting and important results concerning contra-continuity, compactness, S-closedness and strong S-closedness. Recently a new weaker form of this class of functions called contra-semicontinuous functions is introduced and investigated by Dontchev and Noiri [5]. They also introduced the notion of RC-continuity [5] between topological spaces which is weaker than contra-continuity and stronger than B-continuity [35]. Quite recently, the present authors [12] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of RC-continuous functions and contra-continuous functions.

The aim of this paper is to introduce and investigate a new class of functions called contra-precontinuous functions which is weaker than contra-continuous functions. In Section 3, we obtain several basic properties of contra-precontinuous functions. In Section 4, we introduce contra-preclosed graphs and investigate relations between contra-precontinuity and contra-preclosed graphs. In Section 5, we obtain some properties of strongly S-closed spaces and compact spaces. Decompositions of RC-continuity and perfect continuity are also obtained. In the last section, we deal with strong forms of connectedness.

2. Preliminaries

In what follows, spaces X and Y are always topological spaces. $C1(A)$ and $\mathrm{Int}(A)$ designate the closure and interior of A which is a subset of X. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{Int}(C1(A))$ (resp. $A = C1(\mathrm{Int}(A))$).

Definition 2.1. *A subset A of a space X is called*

(i) preopen [16] *if* $A \subset \mathrm{Int}(C1(A))$,

(ii) semi-open [15] *if* $A \subset C1(\mathrm{Int}(A))$,

(iii) α-open [22] *if* $A \subset \mathrm{Int}(C1(\mathrm{Int}(A)))$,

(iv) β-open [1] *if* $A \subset C1(\mathrm{Int}(C1(A)))$.

The complement of a preopen (resp. semi-open, α-open, β-open) set is said to be *preclosed* (resp. *semi-closed*, *α-closed*, *β-closed*). The collection of all closed (resp. preopen, semi-open, α-open and β-open) subsets of X will be donted by $C(X)$ (resp. $PO(X)$, $SO(X)$, $\alpha(X)$ and $\beta(X)$). We set $C(X,x) = \{V \in C(X) \mid x \in V\}$ for $x \in X$. We define similarly $PO(X,x)$, $SO(X,x)$, $\alpha(X,x)$ and $\beta(X,x)$.

Definition 2.2. *A function $f : X \to Y$ is called perfectly continuous* [23] *(resp. RC-continuous* [5]*) if for each open set V of Y, $f^{-1}(V)$ is clopen (resp. regular-closed) in X.*

Definition 2.3. *A function $f : X \to Y$ is called precontinuous* [16] *(resp. semi-continuous* [15], *β-continuous* [1]*) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in PO(X,x)$* (resp. $U \in SO(X,x)$, $U \in \beta(X,x)$) *such that $f(U) \subset V$.*

Definition 2.4. *A function $f : X \to Y$ is called almost precontinuous* [20] *if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists $U \in PO(X,x)$ such that* $f(U) \subset \mathrm{Int}(C1(V))$.

Definition 2.5. *A function $f : X \to Y$ is called contra-precontinuous (resp. contra-continuous* [3], *contra-semicontinuous* [5], *contra-α-continuous* [11], *contra-β-continuous* [3]*) if $f^{-1}(V)$ is preclosed (resp. closed, semi-closed, α-closed, β-closed) in X for each open set V of Y.*

For the functions defined above, we have the following diagram:

$$\begin{tikzcd}
\text{perfectly continuous} \arrow[r, Rightarrow] & RC\text{-continuous} \arrow[r, Rightarrow] & \text{contra-continuous} \arrow[d, Rightarrow] & \\
 & & \text{contra-}\alpha\text{-continuous} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{contra-precontinuous} \arrow[d, Rightarrow] \\
 & & \text{contra-semicontinuous} \arrow[r, Rightarrow] & \text{contra-}\beta\text{-continuous}
\end{tikzcd}$$

Remark 2.1. It should be noticed that contra-precontinuity and precontinuity are independent notions as shown by the following examples due to Dontchev [2].

Example 2.1. A continuous function need not be contra-precontinuous. The identity function on the real line with the usual topology is an example of a continuous function which is not contra-precontinuous.

Example 2.2. A contra-precontinuous function need not be precontinuous. Let $X = \{a,\ b\}$ be the Sierpinski space by setting $\tau = \{\emptyset,\ \{a\},\ X\}$ and $\sigma = \{\emptyset,\ \{b\},\ X\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is contra-precontinuous. But it is neither precontinuous nor semi-continuous.

3. Some properties

Definition 3.1. *Let A be a subset of a space $(X,\ \tau)$.*

(1) *The set* $\bigcap \{U \in \tau \mid A \subset U\}$ *is called the kernel of A* [19] *and is denoted by* $\ker(A)$,

(2) *The set* $\bigcap \{F \in X \mid A \subset F,\ F : \text{preclosed}\}$ *is called the preclosure of A* [7] *and is denoted by* $pC1(A)$.

Lemma 3.1. *The following properties hold for subsets A, B of a space X:*

(1) $x \in \ker(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any* $F \in C(X, x)$.

(2) $A \subset \ker(A)$ *and* $A = \ker(A)$ *if* A *is open in X.*

(3) *If* $A \subset B$, *then* $\ker(A) \subset \ker(B)$.

Theorem 3.1. *The following are equivalent for a function* $f : X \to Y$:

(1) f *is contra-precontinuous;*

(2) *for every closed subset F of Y,* $f^{-1}(F) \in PO(X)$;

(3) *for each* $x \in X$ *and each* $F \in C(Y, f(x))$, *there exists* $U \in PO(X, x)$ *such that* $f(U) \subset F$;

(4) $f(pC1(A)) \subset \ker(f(A))$ *for every subset* A *of X;*

(5) $pC1(f^{-1}(B)) \subset f^{-1}(\ker(B))$ *for every subset of B of Y.*

Proof. The implications $(1) \Leftrightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (2)$: Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in PO(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \bigcup\{U_x \,|\, x \in f^{-1}(F)\} \in PO(X)$.

$(2) \Rightarrow (4)$: Let A be any subset of *X*. Suppose that $y \notin \ker(f(A))$. Then by Lemma 3.1 there exists $F \in C(X, Y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $pC1(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(pC1(A)) \cap F = \emptyset$ and $y \notin f(pC1(A))$. This implies that $f(C1(A)) \subset \ker(f(A))$.

$(4) \Rightarrow (5)$: Let B be any subset of *Y*. By (4) and Lemma 3.1, we have $f(pC1(f^{-1}(B))) \subset \ker(f(f^{-1}(B)) \subset \ker(B)$ and $pC1(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

$(5) \Rightarrow (1)$: Let V be any open set of *Y*. Then, by Lemma 3.1 we have $pC1(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $pC1(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is preclosed in X.

Theorem 3.2. *The following are equivalent for a function* $f : X \to Y$ *:*

(i) f *is contra-*α*-continuous;*

(ii) f *is contra-precontinuous and contra-semicontinuous.*

Proof. This follows from the fact that $A \in \alpha(X)$ if and only if $A \in PO(X) \cap SO(X)$ [28, Lemma 1].

Theorem 3.3. *If a function* $f : X \to Y$ *is contra-precontinuous and Y is regular, then* f *is precontinuous.*

Proof. Let x be an arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $C1(W) \subset V$. Since f is contra-precontinuous, so by Theorem 3.1 there exists $U \in PO(X, x)$ such that $f(U) \subset C1(W)$. Then $f(U) \subset C1(W) \subset V$. Hence, f is precontinuous.

Remark 3.1. By Example 2.1, a precontinuous functions $f : X \to Y$ is not always contra-precontinuous even if Y is regular.

Recall that a function $f : X \to Y$ is called *M-preopen* [17] if the image of each preopen set is preopen.

Theorem 3.4. *If $f : X \to Y$ is an M-preopen contra-precontinuous function, then f is almost precontinuous.*

Proof. Let x be any arbitrary point of X and V be an open neighborhood $f(x)$. Since f is contra-precontinuous, then by Theorem 3.1 (3), there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$. Since f is M-preopen, $f(U)$ is preopen in Y. Therefore $f(U) \subset \mathrm{Int}(C1(f(U))) \subset \mathrm{Int}(C1(V))$. This shows that f is almost precontinuous.

Definition 3.2. *A function $f : X \to Y$ is said to be almost weakly continuous* [13] *if $f^{-1}(V) \subset \mathrm{Int}(C1(f^{-1}(C1(V)))$ for every open set V of Y.*

It is shown in [25, Theorem 3.1] that a function $f : X \to Y$ is almost weakly continuous if and only if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$.

Remark 3.2. The following implications are obvious:

$$\text{precontinuity} \Rightarrow \text{almost precontinuity} \Rightarrow \text{almost weak continuity,}$$

where the converses are false as shown in Examples 2.1 and 2.2 [10].

As shown in Example 2.2, a contra-precontinuous function need not be precontinuous. However, every contra-precontinuous function is necessarily almost weakly continuous.

Theorem 3.5. *If a function $f : X \to Y$ is contra-precontinuous, then f is almost weakly continuous.*

Proof. Let V be any open set of Y. Since $Cl(V)$ is closed in Y, $f^{-1}(Cl(V))$ is preopen in X and we have $f^{-1}(V) \subset f^{-1}(Cl(V)) \subset \mathrm{Int}(Cl(f^{-1}(Cl(V))))$. This shows that f is almost weakly continuous.

The *prefrontier* [26] $\mathrm{pFr}(A)$ of A, where $A \subset X$, is defined by $\mathrm{pFr}(A) = pCl(A) \cap pCl(X - A)$.

Theorem 3.6. *The set of all points of x of X at which $f : X \to Y$ is not contra-precontinuous is identical with the union of the prefrontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. "Necessity". Suppose that f is not contra-precontinuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in PO(X, x)$. This implies that $U \cap f^{-1}(Y - F) \neq \emptyset$. Therefore, we have $x \in pCl(f^{-1}(Y - F)) = pCl(X - f^{-1}(F))$. However, since $x \in f^{-1}(F)$, $x \in pCl(f^{-1}(F))$. Therefore, we obtain $x \in \mathrm{pFr}(f^{-1}(F))$.
"Sufficiency". Suppose that $x \in \mathrm{pFr}(f^{-1}(F))$ for some $F \in C(Y, f(x))$. Now, we assume that f is contra-precontinuous at x. Then there exists $U \in PO(X, x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in p\mathrm{Int}(f^{-1}(F)) \subset X - \mathrm{pFr}(f^{-1}(F))$. This is a contradiction. This means that f is not contra-precontinuous.

Recall that a family E of subsets of a space (X, τ) is called a *network* for a topology τ on (X, τ) if every set in τ is the union of some subfamily of E.

Definition 3.3. *A space (X, τ) is said to be*

(1) *extremally disconnected* [33] *if the closure of every open set of X is open in X,*
(2) *locally indiscrete* [21] *if every open set of X is closed in X.*
(3) *submaximal* [29] *if every dense set of X is open in X, equivalently if every preopen set is open,*
(4) *strongly irresolvable* [8] *if no nonempty open set is resolvable, equivalently if every preopen subset is α-open,*
(5) *mildly Hausdorff* [6] *if the δ-closed sets form a network for its topology τ, where a δ-closed set is the intersection of regular closed sets,*
(6) *strongly S-closed* [3] *if every closed cover of X has a finite subcover,*
(7) *door space* [4] *if every subset of X is either open or closed.*

Remark 3.3. It should be noted that every door space X is submaximal [4, Theorem 2.7] and every mildly Hausdorff strongly S-closed space is locally indiscrete [6].

The following results follow immediately from Definition 3.3 and Remark 3.3:

Theorem 3.7. *If a function $f : X \to Y$ is continuous and X is locally indiscrete, then f is contra-continuous.*

Corollary 3.1. *If a function $f : X \to Y$ is continuous and X is mildly Hausdorff strongly S-closed, then f is contra-continuous.*

Theorem 3.8. *Let $f : X \to Y$ be a contra-precontinuous function.*

(1) If X is submaximal, then f is contra-continuous,
(2) If X is strongly irresolvable, then f is contra-α-continuous.

Corollary 3.2. *If a function $f : X \to Y$ is contra-precontinuous and X is a door space, then f is contra-continuous.*

Lemma 3.2. *For a subset A of a space X, the following are equivalent:*

(1) A is regular closed;
(2) A is preclosed and semi-open;
(3) A is α-closed and β-open.

Proof. $(1) \Rightarrow (2)$: Let A be regular closed. Then $A = C1(\text{Int}\,(A))$ and A is preclosed and semi-open.

$(2) \Rightarrow (3)$: Let A be preclosed and semi-open. Then $C1(\text{Int}(A)) \subset A$ and $A \subset C1(\text{Int}(A))$. Therefore, we have $C1(\text{Int}(A)) = C1(A)$ and hence $C1(\text{Int}(C1(A))) = C1(\text{Int}(C1(\text{Int}(A)))) = C1(\text{Int}(A)) \subset A$. This shows that A is α-closed. Since $SO(X) \subset \beta O(X)$, it is obvious that A is β-open.

$(3) \Rightarrow (1)$: Let A be α-closed and β-open. Then $A = C1(\text{Int}(C1(A)))$ and hence $C1(\text{Int}(A)) = C1(\text{Int}[C1(\text{Int}(C1(A)))]) = C1(\text{Int}(C1(A))) = A$. Therefore, A is regular closed.

As a consequence of the above lemma, we have the following result:

Theorem 3.9. *The following statements are equivalent for a function $f : X \to Y$:*

(1) f is RC-continuous;
(2) f is contra-precontinuous and semi-continuous;
(3) f is contra-α-continuous and β-continuous.

4. Contra-preclosed graphs

We begin with the following notion:

Definition 4.1. *The graph $G(f)$ of a function $f : X \to Y$ is said to be contra-preclosed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in PO(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 4.1. *The graph $G(f)$ of $f : X \to Y$ is contra-preclosed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in PO(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.*

Theorem 4.1. *If $f : X \to Y$ is contra-precontinuous and Y is Urysohn, then $G(f)$ is contra-preclosed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets V, W such that $f(x) \in V$, $y \in W$ and $C1(V) \cap C1(W) = \emptyset$. Since f is contra-precontinuous, there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$. Therefore, we obtain $f(U) \cap C1(W) = \emptyset$. This shows that $G(f)$ is contra-preclosed.

Definition 4.2. *A space X is said to be*

(1) *strongly compact* [17] *if every preopen cover of X has a finite subcover,*
(2) *S-closed* [34] *if every semi-open cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X, there exists a finite subset $\nabla_\circ$ of ∇ such that $X = \bigcup\{C1(V_\alpha) \mid \alpha \in \nabla_\circ\}$, equivalently if every regular closed cover of X has a finite subcover,*
(3) *mildly compact* [32] *if every clopen cover of X has a finite subcover.*

Definition 4.3. *A subset S of a space X is said to be*

(1) *strongly compact relative to X* [17] *if every cover of S by preopen sets of X has a finite subcover,*
(2) *strongly S-closed* [3] *if the subspace S is strongly S-closed.*

Theorem 4.2. *Let X be submaximal. If $f : X \to Y$ has a contra-preclosed graph, then the inverse image of a strongly S-closed set K of Y is closed in X.*

Proof. Assume that K is a strongly S-closed set of Y and $x \notin f^{-1}(K)$. For each $k \in K$, $(x, k) \notin G(f)$. By Lemma 4.1, there exists $U_k \in PO(X, x)$ and $V_k \in C(Y, k)$ such that $f(U_k) \cap V_k = \emptyset$. Since $\{K \cap V_k \mid k \in K\}$ is a closed cover of the subspace K, there exists a finite subset $K_1 \subset K$ such that

$K \subset \bigcup\{V_k \mid k \in K_1\}$. Set $U = \bigcap\{U_k \mid k \in K_1\}$, then U is open since X is submaximal. Therefore $f(U) \cap K = \emptyset$ and $U \cap f^{-1}(K) = \emptyset$. This shows that $f^{-1}(K)$ is closed in X.

A space X is said to be *weakly Hausdorff* [31] if each point of X is an intersection of regular closed sets of X.

Corollary 4.1. *Let X be submaximal and Y be strongly S-closed weakly Hausdorff. The following properties are equivalent for a function* $f: X \to Y$:

(1) f *is contra-precontinuous;*
(2) $G(f)$ *is contra-preclosed;*
(3) $f^{-1}(K)$ *is closed in X for every strong S-closed set K of Y;*
(4) f *is contra-continuous.*

Proof. $(1) \Rightarrow (2)$: It is shown in [9, Theorem 3.7] that every *S*-closed weakly Hausdorff space is extremally disconnected. Since a strongly *S*-closed space is *S*-closed, Y is extremally disconnected and hence every regular closed set of Y is clopen. This shows that Y is Urysohn. By Theorem 4.1, $G(f)$ is contra-preclosed.
$(2) \Rightarrow (3)$: This is a result of Theorem 4.2.
$(3) \Rightarrow (4)$: First, we show that an open set of Y is strongly *S*-closed. Let V be an open set of Y and $\{H_\alpha \mid \alpha \in \nabla\}$ be a cover of V by closed sets H_α of the subspace V. For each $\alpha \in \nabla$, there exists a closed set K_α of X such that $H_\alpha = K_\alpha \cap V$. Then, the family $\{K_\alpha \mid \alpha \in \nabla\} \cup (Y - V)$ is a closed cover of Y. Since Y is strongly *S*-closed, there exists a finite subset $\nabla_\circ \subset \nabla$ such that $Y = \bigcup\{K_\alpha \mid \alpha \in \nabla_\circ\} \cup (Y - V)$. Therefore we obtain $V = (\bigcup\{K_\alpha \mid \alpha \in \nabla_\circ\}) \cap V = \bigcup\{H_\alpha \mid \alpha \in \nabla_\circ\}$. This shows that V is strongly *S*-closed. For any open set V, by (3) $f^{-1}(V)$ is closed in X and f is contra-continuous.

5. Strong forms of compactness

Theorem 5.1. *If* $f: X \to Y$ *is contra-precontinuous and K is strongly compact relative to X, then* $f(K)$ *is strongly S-closed in Y.*

Proof. Let $\{H_\alpha \mid \alpha \in \nabla\}$ be any cover of $f(K)$ by closed sets of the subspace $f(K)$. For each $\alpha \in \nabla$, there exists a closed set K_α of Y such that $H_\alpha = K_\alpha \cap f(K)$. For each $x \in K$, there exists $\alpha(x) \in \nabla$ such that

$f(x) \in K_{\alpha(x)}$ and by Theorem 3.1 there exists $U_x \in PO(X, x)$ such that $f(U_x) \subset K_{\alpha(x)}$. Since the family $\{U_x \mid x \in K\}$ is a preopen cover of *K*, there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x \mid x \in K_\circ\}$. Therefore, we obtain $f(K) \subset \bigcup\{f(U_x) \mid x \in K_\circ\}$ which is a subset of $\bigcup\{K_{\alpha(x)} \mid \alpha \in K_\circ\}$. Thus, $f(K) = \bigcup\{H_{\alpha(x)} \mid x \in K_\circ\}$ and hence $f(K)$ is strongly *S*-closed.

Corollary 5.1. *If $f : X \to Y$ is contra-precontinuous surjection and X is strongly compact, then Y is strongly S-closed.*

Theorem 5.2. *A function $f : X \to Y$ is RC-continuous if and only if it is contra-precontinuous and semi-continuous.*

Proof. "Necessity". Every *RC*-continuous function is contra-continuous and hence contra-precontinuous. Since every regular closed set is semi-open, *RC*-continuous functions are semi-continuous.
"Sufficiency". For any open set *V* of *Y*, $f^{-1}(V)$ is preclosed and semi-open in X and hence we have $C1(\operatorname{Int}(f^{-1}(V))) \subset f^{-1}(V) \subset C1(\operatorname{Int}(f^{-1}(V)))$. Therefore, we obtain $C1(\operatorname{Int}(f^{-1}(V))) = f^{-1}(V)$ and hence f is *RC*-continuous.

Remark 5.1. It follows from Examples 2.1 and 2.2 that contra-precontinuity and semi-continuity are independent of each other. Therefore, by Theorem 5.2 we had a decomposition of *RC*-continuity.

Theorem 5.3. *If $f : X \to Y$ is an RC-continuous surjection and X is S-closed, then Y is compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of *Y*. Then $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_\circ\}$ is a regular closed cover of *X* and we have $X = \bigcup\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_\circ\}$ for some finite subset $\nabla_\circ$ of ∇. Since f is surjective, $Y = \bigcup\{V_\alpha \mid \alpha \in \nabla_\circ\}$ and Y is compact.

A function $f : X \to Y$ is said to be *α-continuous* [18] if $f^{-1}(V) \in \alpha(X)$ for every open set V of *Y*. In [2, Theorem 2.9], Dontchev obtained decompositions of perfect continuity. The following is also a decomposition of perfect continuity.

Theorem 5.4. *A function $f : X \to Y$ is perfectly continuous if and only if it is contra-precontinuous and α-continuous.*

Proof. "Necessity". This is obvious.
"Sufficiency". Let f be contra-precontinuous and α-continuous. Let V be any open set of Y. Then $f^{-1}(V)$ is preclosed and α-open in X. Therefore, we have

$$\begin{aligned}\mathrm{Int}(C1(\mathrm{Int}(f^{-1}(V)))) &\subset C1(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(V) \\ &\subset \mathrm{Int}(C1(\mathrm{Int}(f^{-1}(V)))) \subset C1(\mathrm{Int}(f^{-1}(V))).\end{aligned}$$

This implies that $f^{-1}(V)$ is clopen in X. Thus, f is perfectly continuous.

Remark 5.2. Contra-precontinuity and α-continuity are independent of each other as shown by Examples 2.1 and 2.2.

Corollary 5.2. *(Dontchev* [3]*). For a functon $f : X \to Y$ the following are equivalent:*

(1) f is perfectly continuous;
(2) f is conitnuous and contra-continuous;
(3) f is α-continuous and contra-continuous.

Theorem 5.5. If *$f : X \to Y$ is a perfectly continuous surjection and X is mildly compact, then Y is compact.*

Proof. Let $f : X \to Y$ be a perfectly continuous surjection and X be mildly compact. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of Y. Then $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a clopen cover of X. Since X is mildly compact, there exists a finite subset $\nabla_\circ$ of ∇ such that $X = \bigcup\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_\circ\}$. Since f is surjective, $Y = \bigcup\{V_\alpha \mid \alpha \in \nabla_\circ\}$ and Y is compact.

A space X is said to be *almost compact* [30] or *quasi H-closed* [27] if every open cover has a finite subfamily the closures of whose members cover X. We have the following implications:

strongly S-closed $\Rightarrow$ S-closed $\Rightarrow$ almost compact $\Rightarrow$ mildly compact.

Corollary 5.3. *(Dontchev* [3]*). The image of an almost compact space under contra-continuous nearly continuous (precontinuous) function is compact.*

Proof. It is shown in [3, Theorem 2.9] that a function is contra-continuous and nearly continuous if and only if it is perfectly continuous. Then, the proof follows from Theorem 5.5.

6. Strong forms of connectedness

Definition 6.1. *A space X is said to be*

(1) *hyperconnected* [33] *if $Cl(V) = X$ for every nonempty open set V of X,*

(2) *θ-irreducible* [14] *if every two nonempty regular closed sets intersect,*

(3) *preconnected* [24] *if X cannot be expressed as the union of two nonempty preopen sets.*

Theorem 6.1. *Let X be preconnected and Y be T_1.* If *$f : X \to Y$ is contra-precontinuous, then f is constant.*

Proof. Since Y is T_1-space, $\cup = \{f^{-1}(y) \mid y \in Y\}$ is a disjoint preopen partition of X. If $|\cup| \geq 2$, then X is the union of two nonempty preopen sets. Since X is preconnected, $|\cup| = 1$. Therefore, f is constant.

A function $f : X \to Y$ is said to be *preclosed* [7] if the image $f(A)$ is preclosed in Y for every closed set A of X.

Theorem 6.2. Let *$f : X \to Y$ be a contra-precontinuous and preclosed surjection. If X is submaximal, then Y is locally indiscrete.*

Proof. Let V be any open set of Y. Since f is contra-continuous and X is submaximal, $f^{-1}(V)$ is closed in X and hence V is preclosed in Y. Therefore, $Cl(V) = Cl(\mathrm{Int}(V)) \subset V$ and V is closed in Y. This shows that Y is locally indiscrete.

Theorem 6.3. *If $f : X \to Y$ is a contra-precontinuous semi-continuous surjection and X is θ-irreducible, then Y is hyperconnected.*

Proof. Suppose that Y is not hyperconnected. Then, there exists two disjoint nonempty open sets V, W and Y. By Theorem 5.2, f is an RC-continuous surjection and $f^{-1}(V)$, $f^{-1}(W)$ are disjoint nonempty regular closed sets of X. Therefore, X is not θ-irreducible.

Theorem 6.4. If *$f : X \to Y$ is a contra-precontinuous α-continuous surjection and X is connected, then Y has an indiscrete topology.*

Proof. Suppose that there exists a proper open set V of Y. By Theorem 5.4, f is a perfectly continuous surjection and $f^{-1}(V)$ is a proper clopen set of X. This shows that X is not connected. Therefore, Y has an indiscrete topology.

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