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More on Semi-Urysohn Spaces

[1]JULIAN DONTCHEV AND [2]MAXIMILIAN GANSTER

[1]Department of Mathematics, University of Helsinki, PL 4, Yliopistonkatu 15, 00014 Helsinki, Finland
[2]Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria

Abstract. The aim of this note is to present some results concerning the class of semi-Urysohn spaces, a concept which has been introduced by M.P. Bhamini [4] under the name of 's-Urysohn spaces'. Semi-Urysohn spaces resp. s-Urysohn spaces have been further investigated in [1], [2] and [5], and quite recently by Noiri and Umehara [20]. Several examples are provided in order to differentiate semi-Urysohn spaces from some other well-known classes of topological spaces. We prove that every Hausdorff space is homeomorphic to a closed subspace of a Hausdorff semi-Urysohn space as well as that the product of every first countable Hausdorff space with the usual space of reals is semi-Urysohn.

1. Preliminaries and some examples

In General Topology, there has been recent interest in the study of some generalizations of Urysohn and related spaces. Weakly Frechet-Urysohn spaces were considered by Malykhin and Tironi [17], anti-Urysohn spaces by Tzannes [27] and semi-Urysohn spaces by Noiri and Umehara [20].

Semi-Urysohn spaces were introduced under the name of s-Urysohn spaces by Bhamini [4], and further investigated by Arya and Bhamini in [1], [2] and [5].

Noiri and Umehara [20] considered this class of spaces in connection with the study of θ-irreducible spaces. A topological space (X,τ) is called *semi-Urysohn* (= s-*Urysohn*) [4] if every two distinct points of X can be separated by semi-open sets whose closures are disjoint. A *semi-open* set is a set that can be placed between an open set and its closure. Complements of semi-open sets are called *semi-closed*.

Recall that a topological space (X,τ) is called *ultra-Hausdorff* [25] if every two distinct points of X can be separated by disjoint clopen sets and *weakly Hausdorff* [24] if each singleton is the intersection of regular closed sets. A topological space is said to be *semi-Hausdorff* [16] if every two distinct points can be separated by disjoint semi-open sets.

A subset A of a space (X,τ) is called *quasi-open* [10] if A is a union of clopen sets. Sets which are open in the semi-regularization topology are usually called δ-*open*. A set A will be called *U-open* if for every $x \in A$ there exists a regular closed set U_1 and a regular open set U_2 such that $x \in U_1 \subseteq U_2 \subseteq A$.

Complements of *U*-open sets will be called *U-closed*. Observe that the following implications hold and none of them is reversible:

$$quasi-open \Rightarrow U-open \Rightarrow \delta-open \Rightarrow open \Rightarrow semi-open$$

Before considering the relationships between semi-Urysohn spaces and some related classes of topological spaces we will point out some new characterizations of this concept. First recall that a set *A* is said to be *semi-regular* if it is both semi-open and semi-closed; it is *sg-open* if its semi-interior contains every semi-closed subset, and β*-open* (or *semi-preopen*) if $A \subseteq cl(\text{int}(cl(A)))$. Those sets are related to each other in the following way. Again, none of the implications is reversible:

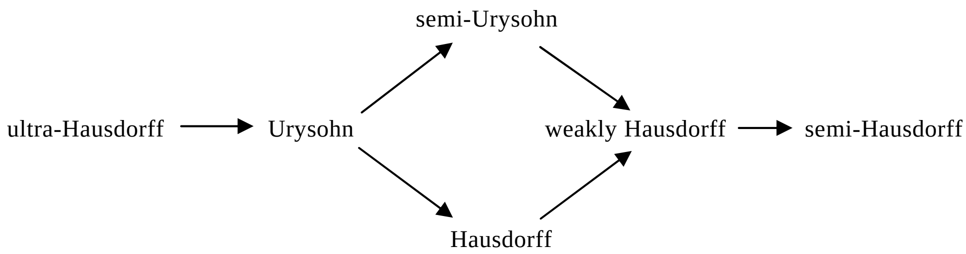
$$semi-regular \Rightarrow semi-open \Rightarrow sg-open \Rightarrow \beta-open$$

Theorem 1.1. *For a topological space* (X,τ) *the following conditions are equivalent:*

(1) X *is semi-Urysohn.*
(2) Every two distinct points can be separated by disjoint regular closed sets.
(3) For every two distinct points x *and* y, *there exist sg-open (or equivalently* β*-open or semi-regular) sets* U *and* V *such that* $x \in U$, $y \in V$ *and* $cl(U) \cap cl(V) = \varnothing$.
(4) Every singleton is U-closed.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follows easily from the fact that the closure of every β-open set is regular closed.
$(2) \Rightarrow (4)$ Let $x \in X$ and let $y \neq x$. By (2) there exist disjoint regular closed sets R and S containing x and y respectively. Note that S and $X \setminus R$ are the required sets from the definition of U-open sets. Thus $X \setminus \{x\}$ is U-open or equivalently $\{x\}$ is U-closed.
$(4) \Rightarrow (2)$ Let $x \neq y$. By (4), $X \setminus \{x\}$ is U-open and hence there exists a regular closed set V and a regular open set U such that $y \in V \subseteq U \subseteq X \setminus \{x\}$. Note that $X \setminus U$ and V are disjoint regular closed sets containing x and y respectively.

The following diagram describes the relationships between semi-Urysohn and some related spaces.



In order to show that the implications in the diagram above are not reversible, we have the following examples.

Example 1.2. The *irrational slope topology* (cf. [26, Example 75]) on the rational upper plane is an example of a countable, connected, first countable Hausdorff space that fails to be semi-Urysohn. Any pair of nonempty regular closed sets has nonempty intersection. This shows that the irrational slope topology is not semi-Urysohn. We also note that the irrational slope topology is *anti-Urysohn* [27], i.e. no two distinct points have disjoint closed neighbourhoods. Additionally, this space is not compact, not even lightly compact. Recall that a space (X,τ) is called *lightly compact* [3] if every locally finite family of nonempty open sets is finite.

Example 1.3. The *scattered unit interval* is an example of a compact, hereditarily semi-regular semi-Urysohn space that fails to be Hausdorff. Let X be the interval $[0,1]$ of real numbers with the following topology: all points from $(0,1)$ are clopen; the basic neighbourhoods of 0 are of the form $[0,x)$ where $x>0$; the neighbourhoods of 1 are of the form $X\setminus F$ where $F\subseteq[0,1)$ is either a finite set or a sequence that converges to 0 with respect to the usual topology. Clearly this space fails to be Hausdorff. It is shown in [21] that the scattered unit interval is compact and hereditarily semi-regular. We need to show that X is semi-Urysohn. For that matter we have to separate 0 and 1 by regular closed sets (all other points are clopen). Set $U=\{\frac{1}{n}: n=2,3\cdots\}$ and $V=(\frac{3}{4},1)$. Observe that $cl(U)$ and $cl(V)$ are the required regular closed sets.

Remark 1.4. We have pointed out that every semi-Urysohn space is weakly Hausdorff. It should also be mentioned that Theorem 2.5.6 in [4] says that every weakly regular weakly Hausdorff space is *s*-Urysohn. For the definition of a weakly regular space see [22].

Remark 1.5. Example 1.2 and Example 1.3 show that the separation axioms 'Hausdorff' and 'semi-Urysohn' are independent from each other. Hausdorff, semi-Urysohn spaces which are not Urysohn exist in profusion.

(1) Consider the 'flat anchor space', i.e. the following subset of the plane: $X = A \cup B \cup C$, where A is the real line, $B = \{(n, 0) : n \in \omega\}$ and $C = \{(1,1), (-1,1)\}$. Every point in the real line is isolated. For each $n \in \omega$, the basic neighborhoods of $(n, 0)$ are the sets $\{(n, 0)\} \cup F$, where F contains all but finitely many points of $(-n-1, n) \cup (n, n+1)$. The basic neighborhoods of $(1,1)$ (resp. $(-1,1)$) have the form $(1,1) \cup (a, \infty)$ (resp. $(-1, -1) \cup (-\infty, a)$) where $a \in A$. This is a semi-regular, Hausdorff space [28]. lt is easily observed that the flat anchor space is semi-Urysohn.

(2) The modified Arens square is another example of a Hausdorff, semi-Urysohn space which is not Urysohn. The underlying set of this space is the subset $A = (0,1) \times (0,1)$ of the Euclidean plane along with the set $B = \{(0, 0), (1, 0)\}$. A base for the topology is defined as follows: we take (1) all open balls $B_r(x)$, where $x \in A$ and $0 < r \in Q$, (2) all sets $U_k(0, 0) = \{(0, 0)\} \cup ((0, \frac{1}{2}) \times (0, \frac{1}{k}))$ for $k \in \omega$ and (3) all sets $U_k(1, 0) = \{(1, 0)\} \cup ((\frac{1}{2}, 1) \times (0, \frac{1}{k}))$ for $k \in \omega$.

2. Basic properties of semi-Urysohn spaces

We start with two fairly general results.

Theorem 2.1. *Let (X, τ) be a Hausdorff space. Then (X, τ) is homeomorphic to a closed subspace of a Hausdorff semi-Urysohn space.*

Proof. Let $Y = X \times \mathsf{R}$ where R denotes the set of reals. We now define a topology an Y. If $x \in X$ and $t \neq 0$, then $\{(x, t)\}$ is open. A basic neighbourhood of $(x, 0)$ has the form $(U \times \mathsf{R}) \setminus F$ where $U \subseteq X$ is open in (X, τ) and F is a finite subset of Y. It is easily seen that Y is Hausdorff and X is homeomorphic to the closed subspace $X \times \{0\}$ of Y. Now let $(x_1, t_1) \neq (x_2, t_2)$. If $t_1 \neq 0$ or $t_2 \neq 0$ we may assume that $t_1 \neq 0$. Then $\{(x_1, t_1)\}$ is clopen in Y and so (x_1, t_1) and (x_2, t_2) can be separated by disjoint regular closed sets. Now suppose that $t_1 = t_2 = 0$. Then $x_1 \neq x_2$. If $V_1 = \{x_1\} \times (\mathsf{R} \setminus \{0\})$ and $V_2 = \{x_2\} \times (\mathsf{R} \setminus \{0\})$ then V_1 and V_2 are open in Y, $\overline{V_1} = \{x_1\} \times \mathsf{R}$ and $\overline{V_2} = \{x_2\} \times \mathsf{R}$, and so $\overline{V_1} \cap \overline{V_2} = \varnothing$. This proves that Y is semi-Urysohn.

Theorem 2.2. *Let* (X, τ) *be a first countable Hausdorff space and let* $\mathbb{R}$ *be the usual space of reals. Then the product space* $Y = X \times \mathbb{R}$ *is semi-Urysohn (and Hausdorff).*

Proof. Let (x_1, t_1) and (x_2, t_2) be two distinct points of $Y = X \times \mathbb{R}$. If $t_1 \neq t_2$, then there exist open intervals $I_1, I_2 \subseteq \mathbb{R}$ containing t_1 resp. t_2 and $\bar{I}_1 \cap \bar{I}_2 = \varnothing$. Then $X \times I_1$ and $X \times I_2$ are open neighbourhoods (in Y) of (x_1, t_1), and (x_2, t_2), respectively, whose closures are disjoint. So let $t_1 = t_2 = t$. Then $x_1 \neq x_2$. Since X is first countable and Hausdorff, choose nested open local bases (in X) $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ at x_1 and x_2, respectively, such that $U_1 \cap V_1 = \varnothing$. Now let $G = \bigcup_{n \in N} (U_n \times (-\infty, t - 1/n))$ and $H = \bigcup_{n \in N} (V_n \times (t + 1/n, \infty))$. Clearly, G and H are open sets and we have $G \cap H = \varnothing$, $\overline{G} \subseteq X \times (-\infty, t]$ and $\overline{H} \subseteq X \times [t, \infty)$, and also $(x_1, t) \in \overline{G} \setminus \overline{H}$ and $(x_2, t) \in \overline{H} \setminus \overline{G}$. Now let us suppose that $(x, s) \in \overline{G} \cap \overline{H}$. Then, $s = t$, $x \neq x_1$ and $x \neq x_2$. Since X is Hausdorff, then there exists $m \in \mathbb{N}$ and an open neighbourhood (in X) $W \subseteq X$ of x such that $W \cap U_n = \varnothing$ and $W \cap V_n = \varnothing$ for each $n > m$. Choose $\varepsilon > 0$ with $\varepsilon < 1/m$. Then $W \times (t - \varepsilon, t + \varepsilon)$ is an open neighbourhood of (x, t). If $(z, r) \in W \times (t - \varepsilon, t + \varepsilon)$, then $z \notin U_n$ for $n > m$, $r > t - 1/m$, and so $(z, r) \notin G$. Hence $(x, t) \notin \overline{G}$. In the same manner we obtain that $(x, t) \notin \overline{H}$, a contradiction. Thus, $\overline{G} \cap \overline{H} = \varnothing$ and we have shown that Y is semi-Urysohn.

An immediate consequence of Theorem 2.1 is:

Corollary 2.3. *A (closed) subspace of a semi-Urysohn space need not be semi-Urysohn.*

We do, however, have the following result. First recall that a subset A of a space (X, τ) is called *locally dense* [7] if $A \subseteq \text{int}(cl(A))$. Note that every open and every dense set is locally dense.

Proposition 2.4. *Locally dense subspaces of semi-Urysohn spaces are semi-Urysohn.*

Proof. Let (X, τ) be semi-Urysohn and let $A \subseteq X$ be locally dense. Let x and y be two distinct points of the subspace $(A, \tau | A)$. By assumption there exist disjoint regular closed subsets U and V of (X, τ) such that $x \in U$ and $y \in V$. Observe that

$A \cap U$ and $A \cap V$ are disjoint regular closed subsets of $(A, \tau|A)$ (cf. [9, Lemma 1.1] and [18, Lemma 4]). By Theorem 1.1, $(A, \tau|A)$ is semi-Urysohn.

Corollary 2.5. *Let $(X_\alpha, \tau_\alpha)_{\alpha\in\Omega}$ be a family of topological spaces. For the topological sum $X = \sum_{\alpha\in\Omega} X_\alpha$ the following conditions are equivalent:*
(1) X is semi-Urysohn.
(2) Each X_α is semi-Urysohn.

Recall that a function $f:(X,\tau)\to(Y,\sigma)$ is called an *R-map* [6] if $f^{-1}(V)$ is regular closed in *X* for every regular closed set *V* of *Y*. *R*-maps are sometimes called rc-continuous functions [14]. A function $f:(X,\tau)\to(Y,\sigma)$ is called *regular closed preserving* [11] if $f(F)$ is regular closed in (Y,σ) for every regular closed subset *F* of (X,τ).

Proposition 2.6. *Let (X,τ) be semi-Urysohn. If $f:(X,\tau)\to(Y,\sigma)$ is a regular closed preserving bijection, then (Y,σ) is also semi-Urysohn.*

Proof. Let *x* and *y* be two distinct points of (Y,σ). Since *f* is bijective, $f^{-1}\{x\}$ and $f^{-1}\{y\}$ are distinct points in (X,τ). Since (X,τ) is a semi-Urysohn space, by Theorem 1.1 there exist disjoint regular closed sets *U* and *V* containing $f^{-1}\{x\}$ and $f^{-1}\{y\}$, respectively. Since *f* is regular closed preserving, $f(U)$ and $f(V)$ are disjoint regular closed subsets of (Y,σ) containing *x* and *y*, respectively. Thus (Y,σ) is also semi-Urysohn.

Since every homeomorphism is a bijective regular closed preserving mapping, we immediately have the following:

Corollary 2.7. *The property 'semi-Urysohn' is a topological property.*

Proposition 2.8. *Let (Y,σ) be semi-Urysohn. If $f:(X,\tau)\to(Y,\sigma)$ is an injective R-map, then (X,τ) is also semi-Urysohn.*

Proof. Similar to the one of Proposition 2.6.

Proposition 2.9. *The product of semi-Urysohn spaces is also a semi-Urysohn space.*

Proof. Follows easily from the facts that (i) Every irresolute (see [8]), almost continuous (see [23]) function is an *R*-map [14] and (ii) the projection map is both irresolute and (almost) continuous.

As a corollary to Theorem 2.2 we observe that a product of two Hausdorff spaces may be semi-Urysohn while not all factor spaces are semi-Urysohn (simply take X to be space considered in Example 1.2).

Recall that a topological space (X,τ) is called a *strongly* T_0-*space* [13] if for every pair of distinct points there exists a regular open set containing one but not the other. lf every regular open subset of X is θ-semi-open [15], i.e. if it is union of regular closed sets, then X is called a *weakly* P_Σ-*space* [19].

Proposition 2.10. *Every strongly* T_0, *weakly* P_Σ *-space is semi-Urysohn.*

We conclude with a further result concerning certain functions between topological spaces. Recall that weakly and strongly θ-irresolute functions were introduced and studied by Ganster, Noiri and Reilly in [12]. A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called *weakly θ-irresolute* (resp. *strongly θ-irresolute)* [12] if the preimage of every regular closed (resp. semi-open) subset of (Y,σ) is semi-open (resp. θ-semi-open) in (X,τ).

Proposition 2.11.

(i) Let (Y,σ) *be semi-Urysohn. If* $f:(X,\tau)\rightarrow(Y,\sigma)$ *is injective and weakly θ-irresolute, then* (X,τ) *is semi-Hausdorff.*

(ii) Let (Y,σ) *be semi-Hausdorff. If* $f:(X,\tau)\rightarrow(Y,\sigma)$ *is injective and strongly θ-irresolute, then* (X,τ) *is semi-Urysohn.*

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