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On Sums of Range Symmetric Matrices in Minkowski Space

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Abstract. We give necessary and sufficient condition for the sums of range symmetric matrices to be range symmetric in Minkowski space $\mathfrak{m}$. As an application it is shown that the sum and parallel sum of parallel summable range symmetric matrices are range symmetric.

1. Introduction

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Throughout we shall deal with $C^{n\times n}$, the space of $n\times n$ complex matrices. Let C^n be the space of complex n-tuples. We shall index the components of a complex vector in C^n from 0 to $n-1$. That is $u = (u_0, u_1, u_2, \cdots, u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, \cdots, -u_{n-1})$. Clearly the Minkowski metric matrix $G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}$ and $G^2 = I_n$. Minkowski inner product on C^n is defined by $(u, v) = \langle u, Gv \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space denoted as $\mathfrak{m}$. With respect to the Minkowski inner product the adjoint of a matrix $A \in C^{n\times n}$ is given by $A^{\sim} = GA^{*}G$, where A^{*} is the usual Hermitian adjoint. Naturally we call a matrix $A \in C^{n\times n}$ $\mathfrak{m}$-symmetric in Minkowski space if $A^{\sim} = A$, and $\mathfrak{m}$-orthogonal if $A^{\sim}A = I$. As in unitary space $\mathfrak{m}$-orthogonal matrices form a group. For $A \in C^{n\times n}$, let $rk(A)$, $R(A)$ and $N(A)$ denote the rank, range space and null space of A respectively.

Definition 1.1. *A^g is said to be a generalized inverse (g-inverse) of A, if* (1.1) $AA^gA = A$.

Definition 1.2. A^r *is said to be a reflexive g inverse of* A *if* (1.2) $AA^rA = A$ *and* $A^rAA^r = A^r$.

Definition 1.3. A^n *is a right (left) normalized g-inverse of* A *if* (1.3) $AA^n = A$, $A^nAA^n = A^n$ *and* AA^n *is* m*-symmetric (* A^nA *is* m*-symmetric).*

Definition 1.4. A^M *is the Minkowski inverse of* A *if* (1.4) $AA^MA = A$, $A^MAA^M = A^M$, AA^M *and* A^MA *are* m*-symmetric*.

In the sequel we shall repeatedly used the following results.

Theorem 1.5. (Theorem 2.2 of [6]) *For* $A \in C^{n\times n}$, *the following are equivalent*

(1) *A is range symmetric in* m
(2) *GA is EP*
(3) *AG is EP*
(4) $N(A^*) = N(AG)$
(5) $R(A) = R(A^{\sim})$
(6) $A^{\sim} = HA = AK$ *for some non-singular matrices H and K.*
(7) $R(A^*) = R(GA)$.

It is well known that in [8] for $A \in C^{n\times n}$, solution exits for equations (1.1) and (1.2). In unitary space for $A \in C^{n\times n}$, since $rk(A) = rk(AA^*) = rk(A^*A)$ solution exists for equation (1.3) and unique solution exists for equation (1.4) which is called the Moore Penrose inverse of A[8]. However this fails in Minkowski space m, since $rk(A) \neq (A^{\sim}A) \neq rk(AA^{\sim})$. In [5] equivalent conditions for the existence of Minkowski inverse for $A \in C^{n\times n}$ has been obtained.

2. Range symmetric matrices

A matrix $A \in C^{n\times n}$ is said to be range symmetric in unitary space (or) equivalently *A* is said to be *EP* if $N(A) = N(A^*)$ (p.163 [1]). For further properties of *EP* matrices one may refer [1, 2 & 7]. In [6], the concept of a range symmetric matrix in m is introduced and developed. In this paper, conditions are obtained for sums of range symmetric matrices in m to be range symmetric in m.

It is shown that the sum and parallel sum of parallel summable range symmetric matrices in m is range symmetric in m for $A \in C^{n\times n}$, $x, y \in C^n$, the Minkowski inner product

$$\begin{aligned}(Ax, y) &= \left\langle Ax, Gy \right\rangle \\ &= \left\langle x, A^* Gy \right\rangle \\ &= \left\langle x, G(GA^*G)\, y \right\rangle \\ &= (x, A^{\sim} y)\end{aligned}$$

$A^{\sim} = GA^*G$ is the Minkowski adjoint of A.

Definition 2.1. $A \in C^{n\times n}$ *is range symmetric in* m *iff* $N(A) = N(A^{\sim})$.

Lemma 2.2. *Let* $A_1, A_2, \cdots, A_m \in C^{n\times n}$. *If* $A = \sum_{i=1}^{m} A_i$ *then* $A = \sum_{i=1}^{m} A_i^{\sim}$.

Proof. By Definition $A_i^{\sim} = GA_i^*G$, for $i = 1, 2, \cdots, m$, where G is Minkowski tensor of order n. To prove $A^{\sim} = \sum_{i=1}^{m} A_i^{\sim}$.

Given $$A = \sum_{i=1}^{m} A_i$$

$$\begin{aligned}\therefore\ A^{\sim} &= G(A_1 + A_2 + \cdots + A_m)^* G \\ &= G(A_1^* + A_2^* + \cdots + A_m^*)G \\ &= A_1^{\sim} + A_2^{\sim} + \cdots + A_m^{\sim} \\ A^{\sim} &= \sum_{i=1}^{m} A_i^{\sim}.\end{aligned}$$

Lemma 2.3. *Let* $A_1, A_2 \in C^{n\times n}$, *then*

(i) $(A_1 A_2)^{\sim} = A_2^{\sim} A_1^{\sim}$ *and*

(ii) $(A_1^{\sim})^{\sim} = A_1$.

Proof. By Definition
$$\begin{aligned}(A_1 A_2)^{\sim} &= G(A_1A_2)^* G \\ &= G(A_2^* A_1^*)G \\ &= (GA_2^*G)(GA_1^*G) \qquad \therefore\ G^2 = I_n \\ &= A_2^{\sim} A_1^{\sim}\end{aligned}$$

(ii) follows from (i).

In the sequel, we shall make use of the following result obtained in [4].

Lemma 2.4. *Let* $A_1, A_2, \cdots, A_m \in C^{n\times n}$ *and let* $A = \sum_{i=1}^{m} A_i$ *consider the following conditions.*

(a) $N(A) \subseteq N(A_i);\ i = 1, 2, \cdots, m.$

(b) $N(A) = \bigcap_{i=1}^{m} N(A_i)$

(c) $rk(A) = rk\begin{bmatrix} A_1 \\ A_2 \\ A_m \end{bmatrix}$

(d) $\sum_{i=1}^{m}\sum_{j=1}^{n} A_1^{*} A_j = 0$

(e) $rk(a) = \sum_{i=1}^{m} rk(A_i).$

Then the following statement hold:

(i) *conditions (a), (b), (c) are equivalent*
(ii) *conditions (d) implies (a) but not the converse.*
(iii) *conditions (e) implies (a) but not the converse.*

Theorem 2.5. *Let* $A_i (i = 1, 2, \cdots, m)$ *be range symmetric in* m*. If any one of the conditions of Lemma* 2.4 *holds, then* $A = \sum_{i=1}^{m} A_i$ *is range symmetric in* m.

Proof. Since each A_i is range symmetric in m, by Definition 2.1, $N(A_i) = N(A_i^{\sim})$ for each $i = 1, 2, \cdots, m$. By the given condition

$$N(A) \subseteq N(A_i),$$

we get

$$N(A) \subseteq \bigcap_{i=1}^{m} N(A_i) = \bigcap_{i=1}^{m} N(A_i^{\sim}).$$

Now,

$$x \in \bigcap_{i=1}^{m} N(A_i^{\sim}) \Rightarrow x \in N(A_i^{\sim}), \text{ for } i = 1 \text{ to } m.$$

$$\Rightarrow A_i^{\sim} x = 0, \text{ for } i = 1 \text{ to } m.$$

$$\Rightarrow (A_1^{\sim} + A_2^{\sim} + \cdots + A_m^{\sim})\, x = 0.$$

$$\Rightarrow A^{\sim} x = 0 \qquad \text{(By Lemma 2.2)}$$

$$\bigcap_{i=1}^{m} N(A_i^{\sim}) \subseteq N(A^{\sim})$$

$N(A) \subseteq \bigcap_{i=1}^{m} N(A_i^{\sim}) \subseteq N(A^{\sim})$ and $rk(A) = rk(A^{\sim})$ implies

$N(A) = N(A^{\sim})$. Thus $A = \bigcap_{i=1}^{n} A_i$ is range symmetric in m.

Remark 2.7. The converse of the Theorem 2.5 is not true. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are range symmetric in m but $N(A + B) \not\subseteq N(A)$.

If A and B are range symmetric in m by Theorem 1.5; $A^{\sim} = H_1 A$ and $B^{\sim} = H_2 B$, where H_1, H_2 are non-singular $n \times n$ matrices. If $H_1 = H_2$ then $(A + B)^{\sim} = A^{\sim} + B^{\sim} = H_1(A + B)$. Again by Theorem 1.5. $A + B$ is range symmetric in m. If $H_1 - H_2$ is non-singular then the above conditions are also necessary for the sum of range symmetric to be range symmetric in m.

Theorem 2.8. *Let A and B be range symmetric in* m $A^{\sim} = H_1 A$ *and* $B^{\sim} = H_2 B$ *such that* $H_1 - H_2$ *is a non-singular matrix. Then* $A + B$ *is range symmetric iff* $N(A + B) \subseteq N(B)$.

Proof. Since $A^{\sim} = H_1 A$ and $B^{\sim} = H_2 B$, A and B are range symmetric follows from Theorem 1.5. Since $N(A + B) \subseteq N(B)$ we can see that $N(A + B) \subseteq N(A)$. Hence by Theorem 2.5, $A + B$ is range symmetric in m.

Conversely, let us assume that $A + B$ is range symmetric in m. Now by Theorem 1.5, $(A + B)^{\sim} = A^{\sim} + B^{\sim} = H_1 A + H_2 B$

$$H(A+B) = H_1A + H_2B$$
$$(H_1 - H)A = (H - H_2)B$$
$$TA = LB$$

where, $T = H_1 - H$ and $L = H - H_2$ such that $T + L = H_1 - H_2$.

$$TA + LA = LB + LA$$
$$(T + L)A = L(A + B)$$

By hypothesis $T + L = H_1 - H_2$ is non-singular. $N(A+B) \subseteq N[L(A+B)] = N[(T+L)A] = N(A)$. Similarly we can see that $N(A+B) \subseteq N(B)$. Thus $A + B$ is range symmetric in $\mathfrak{m}$ implies $N(A+B) \subseteq N(A)$ and $N(B)$. Hence the Theorem.

3. Parallel summable range symmetric matrices

In this section we shall show that the sum and parallel sum of parallel summable range symmetric matrices in $\mathfrak{m}$ are range symmetric. First we shall give the Definitions and some properties of parallel summable (p.s) matrices (p.188 [9]).

Definition 3.1. *For complex matrices A and B are said to be p.s in unitary space if* $N(A+B) \subseteq N(B)$ *and* $N(A+B)^* \subseteq N(B^*)$ *(or) Equivalently* $N(A+B) \subseteq N(A)$ *and* $N(A+B)^* \subseteq N(A^*)$.

Definition 3.2. *If A and B are p.s then parallel sum of A and B denoted by* $A : B$ *is defined as* $A : B = A(A+B)^- B$.

[The product $A(A+B)^- B$ is invariant for all choices of generalized inverse $(A+B)^-$ of $A + B$ under the conditions that A and B are p.s. (p.21 [9])].

In general for any $A \in C^{n\times n}$, $N(A^*) \neq N(A^\sim)$ for instance let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; A^\sim = GA^*G = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $N(A^*) = \left\{ y : y = \begin{pmatrix} x \\ -x \end{pmatrix} \right\}$; $N(A^\sim) = \left\{ y : y = \begin{pmatrix} x \\ x \end{pmatrix} \right\}$. Therefore $N(A^*) \neq N(A^\sim)$.

Lemma 3.3. *Let A and B be matrices in* m. *Then* $N(A^{*}) \subseteq N(B^{*})$ *iff* $N(A^{\sim}) \subseteq N(B^{\sim})$.

Proof. Let us assume that $N(A^{*}) \subseteq N(B^{*})$. We need to prove $N(A^{\sim}) \subseteq N(B^{\sim})$. Let us choose

$$\begin{aligned} x \in N(A^{\sim}) &\Rightarrow A^{\sim}x = 0 \\ &\Rightarrow GA^{*}Gx = 0 \\ &\Rightarrow A^{*}Gx = 0 \\ &\Rightarrow A^{*}y = 0 \quad \text{where} \quad y = Gx \\ &\Rightarrow y \in N(A^{*}) \subseteq N(B^{*}) \\ &\Rightarrow B^{*}y = 0 \\ &\Rightarrow B^{*}Gx = 0 \\ &\Rightarrow B^{\sim}x = 0 \\ &\Rightarrow x \in N(B^{\sim}). \text{ Thus } \quad N(A^{\sim}) \subseteq N(B^{\sim}) \end{aligned}$$

Conversely let us assume that $N(A^{\sim}) \subseteq N(B^{\sim})$. We need to show that $N(A^{*}) \subseteq N(B^{*})$. Let us choose

$$\begin{aligned} x \in N(A^{*}) &\Rightarrow A^{*}x = 0 \\ &\Rightarrow (GA^{*}G)Gx = 0 \qquad [\text{By } G^2 = In] \\ &\Rightarrow A^{\sim}Gx = 0 \\ &\Rightarrow A^{\sim}y = 0 \qquad \text{where } y = Gx \\ &\Rightarrow y \in N(A^{\sim}) \subseteq N(B^{\sim}) \\ &\Rightarrow B^{\sim}y = 0 \\ &\Rightarrow GB^{*}GGx = 0 \\ &\Rightarrow GB^{*}x = 0 \\ &\Rightarrow B^{*}x = 0 \\ &\Rightarrow x \in N(B^{*}). \end{aligned}$$

Thus $N(A^{*}) \subseteq N(B^{*})$. Hence the Lemma.

By using Lemma 3.3, Definition 3.1 can be modified as follows.

Definition 3.1′. *A pair of matrices A and B are said to be p.s. in* m *if* $N(A+B) \subseteq N(B)$ *and* $N(A+B)^{\sim} \subseteq N(B^{\sim})$ *or equivalently* $N(A+B) \subseteq N(A)$ *and* $N(A+B)^{\sim} \subseteq N(A^{\sim})$.

Properties. Let A and B a pair of p.s. matrices in m. Then the following holds.

P.1 : $A:B$

P.2 : $A^{\sim}$ and $B^{\sim}$ are p.s. in m and $(A:B)^{\sim} = A^{\sim} : B^{\sim}$.

P.3 : If U is non-singular then UA and UB are p.s. and $UA:UB = U(A:B)$

P.4 : $N(A:B) = N(A) + N(B)$

Proof. P.1, P.3 and P.4, have been proved in (P.188 [9]). Here we shall prove only P.2, A and B are p.s. in m. $A^{\sim}$ and $B^{\sim}$ are p.s. in m follows from Lemma 2.2 and Lemma 2.3.

$$
\begin{aligned}
A^{\sim} : B^{\sim} &= A^{\sim}(A^{\sim} + B^{\sim})^{\sim} B^{\sim} && \text{[By Definition 3.2]} \\
&= A^{\sim}[(A + B)^{\sim}]^{-} B^{\sim} && \text{[By Definition 2.2]} \\
&= A^{\sim}[(A + B)^{-}]^{\sim} B^{\sim} && \\
&= [B(A + B)^{-} A]^{\sim} && \text{[By Lemma 2.3]} \\
&= [A(A + B)^{-} B]^{\sim} && \text{[By P.1]} \\
&= [A:B]^{\sim} &&
\end{aligned}
$$

Lemma 3.4. *Let A and B be range symmetric in* m. *Then A and B are p.s. range symmetric in* m *iff* $N(A + B) \subseteq N(A)$.

Proof. A and B are parallel summable, by Definition 3.1, it follows that $N(A + B) \subseteq N(A)$.

Conversely, if $N(A + B) \subseteq N(A)$, then $N(A + B) \subseteq N(B)$. Since A and B are range symmetric in m. $A + B$ is range symmetric in m by Theorem 2.5. Hence $N(A + B) = N(A + B)^{\sim}$ and $N(A + B) \subseteq N(A)$ implies $N(A + B)^{\sim} \subseteq N(A^{\sim})$. Therefore by Definition 3.1′ A and B are p.s. range symmetric in m. Hence the Lemma.

Remark 3.5. Lemma 3.4 fails if we relax the condition that A and B are range symmetric in m. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ be range symmetric in m, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not range symmetric in $\mathfrak{m}$. $A^{\sim} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B^{\sim} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $(A + B)^{\sim} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. $N(A + B) \subseteq N(A)$ and $N(B)$ but $N(A + B)^{\sim} \not\subseteq N(A^{\sim})$ and $N(B^{\sim})$. Hence A and B are not p.s.

Theorem 3.6. *Let A and B be p.s. range symmetric in* $\mathfrak{m}$. *Then* $A : B$ *and* $A + B$ *are range symmetric in* $\mathfrak{m}$.

Proof. Since A and B are p.s. range symmetric in $\mathfrak{m}$. $N(A + B) \subseteq N(A)$ and $N(A + B) \subseteq N(B)$, follows from Lemma 3.4. Now the fact that $A + B$ is range symmetric in $\mathfrak{m}$ follows from Theorem 2.5. $A : B$ is range symmetric in $\mathfrak{m}$ runs as follows.

$$
\begin{aligned}
N(A:B)^{\sim} &= N(A^{\sim} : B^{\sim}) && \text{[By P.2]} \\
&= N(A^{\sim}) + N(B^{\sim}) && \text{[By P.4]} \\
&= N(A) + N(B) && \text{[By definition 2.1]} \\
&= N(A:B) && \text{[By P.4]}
\end{aligned}
$$

Thus $A : B$ is range symmetric in $\mathfrak{m}$ whenever A and B are p.s. range symmetric in $\mathfrak{m}$. Hence the Theorem.

Definition 3.7. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be an* $n \times n$ *matrix. The schur complement of* A *in M, denoted by M/A is defined as* $D - CA^{-}B$, *where* A^{-} *is a generalized inverse of* A *(p.*291 [3]*).*

Theorem 3.8. *Let A and B be range symmetric in* $\mathfrak{m}$ *of rank* r_1 *and* r_2 *respectively, such that* $N(A + B) \subseteq N(B)$. *Then there exists a* $2n \times 2n$ *range symmetric matrix M of rank r, with schur complement of C in M is EP, where* $r_1 + r_2 = r$, $C = A + B$.

Proof. Since A and B are range symmetric in $\mathfrak{m}$, by Theorem 1.5 GA and GB are EPr_1 and EPr_2 matrices. By Theorem 1 [7] there exists unitary matrices U and V of order n, such that $GA = U^{*}DU$ and $GB = V^{*}EV$, where

$$D = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}, \; H \text{ is } r_1 \times r_1, \text{ non-singular matrix}$$

$$E = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}, \; K \text{ is } r_2 \times r_2, \text{ non-singular matrix}$$

Let us define $P = \begin{bmatrix} V & 0 \\ U & I \end{bmatrix}$.

Now
$$P^* \begin{bmatrix} E & 0 \\ 0 & D \end{bmatrix} P = \begin{bmatrix} V^* & U^* \\ 0 & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} V & 0 \\ U & I \end{bmatrix}$$
$$= \begin{bmatrix} V^*E & U^*D \\ 0 & D \end{bmatrix} \begin{bmatrix} V & 0 \\ U & I \end{bmatrix}$$
$$= \begin{bmatrix} V^*EV + U^*DU & GAU^* \\ UGA & UGAU^* \end{bmatrix}$$
$$= \begin{bmatrix} GA + GB & GAU^* \\ UGA & UGAU^* \end{bmatrix} = \overline{M}$$

$\overline{M}$ is $2n \times 2n$ matrix, $rk(\overline{M}) = rk\, D + rk\, E = r_1 + r_2 = r$.

Let us define $Q = \begin{bmatrix} G & 0 \\ -U & -I_n \end{bmatrix}$, Q is non-singular matrix.

Since A and B are range symmetric in m and GA, BG are EP by Theorem 1.5 and $UGAU^*$ is EP [2], we can express $\overline{M}$ as follows.

$$\overline{M} = Q^* \begin{bmatrix} BG & 0 \\ 0 & UGAU^* \end{bmatrix} Q$$
$$= \begin{bmatrix} G & -U^* \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} BG & 0 \\ 0 & UGAU^* \end{bmatrix} \begin{bmatrix} G & 0 \\ -U & -I_n \end{bmatrix}$$
$$= \begin{bmatrix} GBG & -GAU^* \\ 0 & -UGAU^* \end{bmatrix} \begin{bmatrix} G & 0 \\ -U & -I_n \end{bmatrix}$$
$$\overline{M} = \begin{bmatrix} GB + GA & GAU^* \\ UGA & UGAU^* \end{bmatrix} = \begin{bmatrix} G(A + B) & GAU^* \\ UGA & UGAU^* \end{bmatrix} \tag{3.1}$$

Since BG, $UGAU^*$ are EP, Q is non-singular, $\overline{M}$ is EP. Since $\overline{M}$ is of $rk\ r$, $\overline{M}$ is EPr. Thus we have proved the existence of the EP matrix $\overline{M}$. Define $M = \overline{G}\,\overline{M}$ where $\overline{G}$ is Minkowski tensor of order $2n$.

$$M = \overline{G}\overline{M} = \begin{bmatrix} G & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} G(A+B) & GAU^* \\ UGA & UGAU^* \end{bmatrix}$$

$$M = \begin{bmatrix} A+B & AU^* \\ -UGA & -UGAU^* \end{bmatrix} = \begin{bmatrix} C & AU^* \\ -UGA & -UGAU^* \end{bmatrix} \qquad (3.2)$$

By Theorem 1.5 M is range symmetric in m of rank r. The schur complement of C in M is

$$\begin{aligned} M/C &= -UGAU^* + UGA\,(A+B)^- AU^* \\ &= -UGAU^* + U(GA+GB)\,(A+B)^- AU^* - UGB\,(A+B)^- AU^* \\ &= -UGAU^* + UGAU^* - UGB(A+B)^- AU^* \\ &= -UG(B:A)U^* && \text{[By Definition 3.2]} \\ &= -UG\,(A:B)U^* && \text{[By P.1]} \\ &= -U(GA:GB)U^* && \text{[By P.3]} \end{aligned}$$

Since A, B are range symmetric in m, GA, GB are EP by Theorem 1.5. By Theorem 4 of [4] $GA : GB$ is EP. Therefore $-U\,(GA:GB)U^*$ is EP. Hence M/C is EP.

Theorem 3.9. *A and B are range symmetric in* m *satisfying the conditions of Theorem* 3.8, *then* $M/C + \overline{G}\,M/GC$, *where G,* $\overline{G}$ *are Minkowski tensor of order n and* $2n$ *respectively*.

Proof. From (3.1), the schur complement of GC in $\overline{M}$ is

$$\begin{aligned} \overline{M}/GC &= UGAU^* - UGA[G(A+B)^-]\,GAU^* \\ &= UGAU^* - U(GA+GB)\,[G(A+B)]^- GAU^* + UGB[G(A+B)]^- GAU^* \\ &= UGAU^* - UGAU^* + UGB(A+B)^- AU^* \\ &= UG\,(B:A)\,U^* && \text{[By Definition 3.2]} \\ &= UG(A:B)U^* && \text{[By P.1]} \\ &= U(GA:GB)U^* && \text{[By P.3]} \end{aligned}$$

Since $M = \overline{G}\,\overline{M}$, $\overline{M} = \overline{G}M$. From (3.2) $M/C = -U(GA:GB)U^*$.

Now $\overline{M}/GC = \overline{G}M/GC = U(GA:GB)U^*$.

Hence $M/C + \overline{G}M/GV = 0$.

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