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Nondegenerate Antisymmetrical Structures in the Bundle of Accelerations

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Abstract. In the present paper we determine all general almost symplectic *N*-linear connections in the bundle of accelerations, we study the group of transformations of these connections and its invariants.

1. Introduction

The differential geometry of higher order Lagrange spaces is introduced and studied by Miron and Atanasiu in [6] – [9].

The various applications of the Lagrange geometry of order k in physics and mechanics are considerable [2], [12].

The study of higher order Lagrange spaces is grounded on the k-osculator bundle notion. The bundle space of accelerations correspond in this study to $k = 2$ [1], [5].

In the present paper we define the notion of general almost symplectic *N*-linear connection (Section 2), the structure of the set of all general almost symplectic *N*-linear connections is discussed (Section 3), and the group of their transformations preserving a nonlinear connection gives us the various important invariants (Section 4).

As to the terminology and notations we use those from [10], which are essentially based on Matsumoto's book [4].

2. The notion of general almost symplectic *N*-linear connection in the bundle of accelerations

Let M be a real C^∞-differentiable manifold of dimension $n(n = 2n')$ and $(Osc^2 M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on the local space $E = Osc^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. If N is a nonlinear connection on E with the coefficients $\underset{(1)}{N}{}^i{}_j$, $\underset{(2)}{N}{}^i{}_j$, then let $D\Gamma(N)=\left(L^i_{jk}\,,\ \underset{(1)}{C^i_{jk}}\,,\ \underset{(2)}{C^i_{jk}}\right)$ be an N-linear connection D on E.

Definition 2.1. *A d-tensor field* $a_{ij}(x,y^{(1)},y^{(2)})$ *of type* (0,2) *on E is called an almost symplectic d-structure on E, if it is alternate and nondegenerate:*

$$a_{ij}\left(x,y^{(1)},y^{(2)}\right)=-a_{ji}\left(x,y^{(1)},y^{(2)}\right) \tag{2.1}$$

$$\det\left(a_{ij}(x,y^{(1)},y^{(2)})\right)\neq 0,\ \forall\ y^{(1)}\neq 0,\ \forall\ y^{(2)}\neq 0. \tag{2.2}$$

Given an almost symplectic d-structure a_{ij}, *we associate Obata's operators:*

$$\Phi^{ir}_{sj}=\frac{1}{2}\left(\delta^i_s\,\delta^r_j-a_{sj}a^{ir}\right),\ \Phi^{*ir}_{sj}=\frac{1}{2}\left(\delta^i_s\,\delta^r_j+a_{sj}a^{ir}\right) \tag{2.3}$$

where (a^{ij}) *is the inverse matrix of* (a_{ij}) *:*

$$a_{ij}a^{jk}=\delta^k_i\ , \tag{2.4}$$

a^{ij} *is also alternate. Obata's operators have the same properties as the ones associated with a Finsler space* [10].

Definition 2.2. *An N-linear connection* $D\Gamma(N)=\left(L^i_{jk},\underset{(1)}{C^i_{jk}},\underset{(2)}{C^i_{jk}}\right)$ *is called a general almost symplectic N-linear connection, if we have:*

$$a_{ij|k}=K_{ijk}\quad,\quad a_{ij}\overset{(\alpha)}{\Big|}_k=\underset{(\alpha)}{Q_{ijk}},\quad(\alpha=1,\,2), \tag{2.5}$$

where K_{ijk}, $\underset{(\alpha)}{Q_{ijk}}$, $(\alpha=1,\,2)$, *are a priori given d-tensor fields of the type* $(0,3)$, *skewsymmetric in the two first indices:*

$$K_{ijk}=-K_{jik},\ \underset{(\alpha)}{Q_{ijk}}=-\underset{(\alpha)}{Q_{jik}}\ ,\ (\alpha=1,\,2). \tag{2.6}$$

Remark. For $K_{ijk}=0$, $\underset{(\alpha)}{Q_{ijk}}\ \ 0$, $(\alpha=1,\,2)$ we obtain the classical case [11], [12].

3. The set of general almost symplectic *N*-linear connections in the bundle of accelerations

Let $D\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{\circ}{C}}{}^i_{jk} \right)$, $(\alpha = 1, 2)$ be a fixed *N*-linear connection on *E*. Then any *N*-linear connection on $E : D\Gamma(N) = \left(L^i_{jk}, \underset{(\alpha)}{C}{}^i_{jk} \right)$, $(\alpha = 1, 2)$ can be expressed in the form:

$$L^i_{jk} = \overset{\circ}{L}{}^i_{jk} - B^i_{jk}, \ \underset{(\alpha)}{C}{}^i_{jk} = \underset{(\alpha)}{\overset{\circ}{C}}{}^i_{jk} - \underset{(\alpha)}{D}{}^i_{jk}, \ (\alpha = 1, 2), \tag{3.1}$$

where $B^i_{jk}, \underset{(\alpha)}{D}{}^i_{jk}$, $(\alpha = 1, 2)$ are components of the difference *d*-tensor fields of $D\Gamma(N)$ from $D\overset{\circ}{\Gamma}(N)$ [4].

In order that $D\Gamma(N)$ is general almost symplectic, that is (2.5) holds for $D\Gamma(N)$, it is necessary and sufficient that $B^i_{jk}, \underset{(\alpha)}{D}{}^i_{jk}$, $(\alpha = 1, 2)$ are satisfy:

$$a_{ij\overset{\circ}{|}k} + a_{sj} B^s_{ik} + a_{is} B^s_{jk} = 0, \ a_{ij} \overset{(\alpha)}{\overset{\circ}{|}}_k + a_{sj} \underset{(\alpha)}{D}{}^s_{ik} + a_{is} \underset{(\alpha)}{D}{}^s_{jk}, \ (\alpha = 1, 2) \tag{3.2}$$

which is equivalent to:

$$\begin{cases} \Phi^{*ir}_{sj} \ B^s_{rk} = \dfrac{1}{2} a^{im} \left(a_{mj\overset{\circ}{|}k} - K_{mjk} \right), \\ \Phi^{*ir}_{sj} \ \underset{(\alpha)}{D}{}^s_{rk} = \dfrac{1}{2} a^{im} \left(a_{mj} \overset{(\alpha)}{\overset{\circ}{|}}_k - \underset{(\alpha)}{Q}{}_{mjk} \right), \quad (\alpha = 1, 2), \end{cases} \tag{3.3}$$

where $\overset{\circ}{|}$ and $\overset{(\alpha)}{\overset{\circ}{|}}$, $(\alpha = 1, 2)$ denote the *h*- and v_α- covariant derivatives $(\alpha = 1, 2)$, with respect to $D\overset{\circ}{\Gamma}(N)$.

Thus, we have:

Proposition 3.1. *Let* $D\overset{\circ}{\Gamma}(N)$ *be a fixed N-linear connection on E. Then the set of all general almost symplectic N-linear connections,* $D\Gamma(N)$, *is given by* (3.1)*, where* B^i_{jk}, $\underset{(\alpha)}{D^i_{jk}}$, $(\alpha = 1, 2)$ *are arbitrary d-tensor fields satisfying* (3.3).

Especially, if $D\overset{\circ}{\Gamma}(N)$ *is general almost symplectic, then* (3.3) *becomes:*

$$\Phi^{*ir}_{sj}\ B^s_{rk} = 0,\ \Phi^{*ir}_{sj}\ \underset{(\alpha)}{D^s_{rk}} = 0,\ (\alpha = 1, 2). \tag{3.4}$$

From Theorem 5.4.3 [13], however, the system (3.3) has solutions in B^i_{jk}, $\underset{(\alpha)}{D^i_{jk}}$, $(\alpha = 1, 2)$. Substituting in (3.1) from the general solution we have:

Theorem 3.1. [14] *Let* $D\overset{\circ}{\Gamma}(N)$ *be a fixed N-linear connection. The set of all general almost symplectic N-linear connections,* $D\Gamma(N)$, *is given by:*

$$\begin{cases} L^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \dfrac{1}{2} a^{im}\left(a_{mj\overset{\circ}{|}k} - K_{mjk}\right) + \Phi^{ir}_{sj}\ X^s_{rk}, \\ \underset{(\alpha)}{C^i_{jk}} = \underset{(\alpha)}{\overset{\circ}{C}{}^i_{jk}} + \dfrac{1}{2} a^{im}\left(a_{mj}\overset{(\alpha)}{\overset{\circ}{|}}{}_k - \underset{(\alpha)}{Q_{mjk}}\right) + \Phi^{ir}_{sj}\ \underset{(\alpha)}{Y^s_{rk}}, \quad (\alpha = 1, 2), \end{cases} \tag{3.5}$$

where X^i_{jk}, $\underset{(\alpha)}{Y^i_{jk}}$, $(\alpha = 1, 2)$ *are arbitrary tensor fields, and* $\overset{\circ}{|}$, $\overset{(\alpha)}{\overset{\circ}{|}}$, *denote the h- and* v_α*- covariant derivatives* $(\alpha = 1, 2)$*, with respect to* $D\overset{\circ}{\Gamma}(N)$.

As the particular case $X^i_{jk} = \underset{(\alpha)}{Y^i_{jk}} = 0$, $(\alpha = 1, 2)$ in Theorem 3.1 we have:

Theorem 3.2. *Let* $D\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{\circ}{C}{}^i_{jk}}\right)$, $(\alpha = 1, 2)$ *be a given N-linear connection. Then the following N-linear connection,* $D\tilde{\Gamma}(N) = \left(\tilde{L}^i_{jk}, \underset{(\alpha)}{\tilde{C}^i_{jk}}\right)$, $(\alpha = 1, 2)$ *is general almost symplectic:*

$$\begin{cases} \tilde{L}^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \dfrac{1}{2} a^{im} \left(a_{m \overset{\circ}{|} k} - K_{mjk} \right) , \\ \underset{(\alpha)}{\tilde{C}^i_{jk}} = \underset{(\alpha)}{\overset{\circ}{C}{}^i_{jk}} + \dfrac{1}{2} a^{im} \left(a_{mj} \overset{(\alpha)}{\overset{\circ}{|}}_k - \underset{(\alpha)}{Q_{mjk}} \right) , \quad (\alpha = 1, 2) , \end{cases} \tag{3.6}$$

where $\overset{\circ}{|}$ *and* $\overset{(\alpha)}{\overset{\circ}{|}}$ *denote the h- and* v_α*- covariant derivatives* $(\alpha = 1, 2)$ *with respect to* $D\overset{\circ}{\Gamma}(N)$.

If we take an almost symplectic *N*-linear connection as $D\overset{\circ}{\Gamma}(N)$ in Theorem 3.2, then (3.6) becomes:

$$\begin{cases} \tilde{L}^i_{jk} = \overset{\circ}{L}{}^i_{jk} - \dfrac{1}{2} a^{im} K_{mjk} , \\ \underset{(\alpha)}{\tilde{C}^i_{jk}} = \underset{(\alpha)}{\overset{\circ}{C}{}^i_{jk}} - \dfrac{1}{2} a^{im} \underset{(\alpha)}{Q_{mjk}} , \quad (\alpha = 1, 2) . \end{cases} \tag{3.7}$$

If we take a general almost symplectic *N*-linear connection (e.g. $D\tilde{\Gamma}(N)$) as $D\overset{\circ}{\Gamma}(N)$ in Theorem 3.1, we have:

Theorem 3.3. *The set of all general almost symplectic N-linear connections* $D\Gamma(N) = \left(L^i_{jk} , \underset{(\alpha)}{C^i_{jk}} \right)$, $(\alpha = 1, 2)$, *is given by:*

$$\begin{cases} L^i_{jk} = \tilde{L}^i_{jk} + \Phi^{ir}_{sj} X^s_{rk} , \\ \underset{(\alpha)}{C^i_{jk}} = \underset{(\alpha)}{\tilde{C}^i_{jk}} + \Phi^{ir}_{sj} \underset{(\alpha)}{Y^s_{rk}} , \quad (\alpha = 1, 2) , \end{cases} \tag{3.8}$$

where X^i_{jk} , $\underset{(\alpha)}{Y^i_{jk}}$, $(\alpha = 1, 2)$ *are arbitrary tensor fields on E.*

4. The group of transformations of the general almost symplectic *N*-linear connections in the bundle of accelerations

Let us consider the transformations $D\Gamma(N) \rightarrow D\tilde{\Gamma}(N)$ of general almost symplectic *N*-linear connections, which preserve the nonlinear connection *N*.

Owing to Theorem 3.3 they are given by:

$$\begin{cases} \overline{L}^i_{jk} = L^i_{jk} + \Phi^{ir}_{sj} X^s_{rk} & , \\ \underset{(\alpha)}{\overline{C}^i_{jk}} = \underset{(\alpha)}{C^i_{jk}} + \Phi^{ir}_{sj} \underset{(\alpha)}{Y^s_{rk}} & , \quad (\alpha = 1, 2), \end{cases} \tag{4.1}$$

where X^i_{jk}, $\underset{(\alpha)}{Y^i_{jk}}$, $(\alpha = 1, 2)$ are arbitrary given tensor fields.

Evidently we have:

Theorem 4.1. *The set of all transformations* (4.1) *and the mapping product from an abelian group* G_{gas}, *which is isomorphic to the additive group of the triads of tensor fields* $\left(\Phi^{ir}_{sj} X^s_{rk}, \Phi^{ir}_{sj} \underset{(1)}{Y^s_{rk}}, \Phi^{ir}_{sj} \underset{(2)}{Y^s_{rk}}\right)$.

We determine the invariants of the G_{gas}. The torsion *d*-tensor fields $\underset{(0)}{T^i_{jk}}$, $\underset{(\alpha)}{S^i_{jk}}$, $\underset{(0\alpha)}{R^i_{jk}}$, $\underset{(\alpha\beta)}{P^i_{jk}}$, $\underset{(\alpha\beta)}{Q^i_{jk}}$, $(\alpha = 1, 2;\ \beta = 1, 2)$ are given in Section 3.4 [12]. We denote with:

$$\underset{(\alpha)}{t^i_{jk}} = \mathcal{A}_{jk} \left\{ \frac{\delta \underset{(\alpha)}{N^i_j}}{\delta y^{(\alpha)k}} \right\}, \quad (\alpha = 1, 2), \tag{4.2}$$

where $\mathcal{A}_{jk}\{\cdots\}$ denotes the alternate summation: $\mathcal{A}_{jk}\{A_{jk}\} = A_{jk} - A_{kj}$.

Since $\underset{(0\alpha)}{R^i_{jk}}$, $\underset{(\alpha)}{t^i_{jk}}$ $(\alpha = 1, 2)$, $\underset{(21)}{Q^i_{jk}}$, $\underset{(12)}{P^i_{jk}}$, $\underset{(21)}{P^i_{jk}}$ depend on *N* only, they are invariants of G_{gas}.

We make some notations:

$$\begin{cases} \underset{(\alpha)}{t^*_{ijk}} = \mathcal{S}_{ijk}\left\{a_{im} \underset{(\alpha)}{t^m_{jk}}\right\} & , \quad \underset{(0)}{T^*_{ijk}} = \mathcal{S}_{ijk}\left\{a_{im} \underset{(0)}{T^m_{jk}}\right\} & , \\ \underset{(0\alpha)}{R^*_{ijk}} = \mathcal{S}_{ijk}\left\{a_{im} \underset{(0\alpha)}{R^m_{jk}}\right\} & , \quad \underset{(\alpha)}{S^*_{ijk}} = \mathcal{S}_{ijk}\left\{a_{im} \underset{(\alpha)}{S^m_{jk}}\right\} & , \quad (\alpha = 1, 2), \end{cases} \tag{4.3}$$

where $\mathcal{S}_{ijk}\{\cdots\}$ denotes the cyclic summation: $\mathcal{S}_{ijk}\{A_{ijk}\} = A_{ijk} + A_{jki} + A_{kij}$, and

$$\begin{cases} \underset{(\alpha\alpha)}{\overset{1}{k}}_{ijk} = a_{km}\underset{(0)}{T}{}^m_{ij} + \mathcal{A}\left\{a_{im}\underset{(\alpha\alpha)}{P}{}^m_{jk}\right\} , \\ \underset{(\alpha)}{\overset{2}{k}}_{ijk} = a_{im}\underset{(\alpha)}{S}{}^m_{jk} + \mathcal{A}_{jk}\left\{a_{km}\underset{(\alpha)}{C}{}^m_{ij}\right\} , \\ \underset{(\alpha\beta)}{\overset{3}{k}}_{ijk} = \mathcal{A}_{jk}\left\{a_{km}\underset{(\alpha\beta)}{P}{}^m_{ij}\right\} , \\ \underset{(\alpha)}{\overset{4}{k}}_{ijk} = \mathcal{A}_{ij}\left\{a_{im}\underset{(\alpha)}{C}{}^m_{jk}\right\} , \quad \alpha = 1, 2; \quad \beta = 1, 2. \end{cases} \tag{4.4}$$

It is noted that $\underset{(\alpha)}{t}{}^*_{ijk}$, $\underset{(0\alpha)}{R}{}^*_{ijk}$, $\underset{(0)}{T}{}^*_{ijk}$, $\underset{(\alpha)}{S}{}^*_{ijk}$ are alternate, $\underset{(\alpha\alpha)}{\overset{1}{k}}_{ijk}$, $\underset{(\alpha)}{\overset{4}{k}}_{ijk}$ are alternate with respect to i, j and $\underset{(\alpha)}{\overset{2}{k}}_{ijk}$, $\underset{(\alpha\beta)}{\overset{3}{k}}_{ijk}$ are alternate with respect to j, k.

By direct calculations we have:

Theorem 4.2. *The d-tensor fields* $\underset{(\alpha)}{t}{}^i_{jk}$, $\underset{(0\alpha)}{R}{}^i_{jk}$, $\underset{(\alpha)}{t}{}^*_{ijk}$, $\underset{(0\alpha)}{R}{}^*_{ijk}$, $\underset{(0)}{T}{}^*_{ijk}$, $\underset{(\alpha)}{S}{}^*_{ijk}$, $\underset{(\alpha\alpha)}{\overset{1}{k}}_{ijk}$, $\underset{(\alpha)}{\overset{2}{k}}_{ijk}$, $\underset{(\alpha\beta)}{\overset{3}{k}}_{ijk}$, $\underset{(\alpha)}{\overset{4}{k}}_{ijk}$, $(\alpha = 1, 2;\ \beta = 1, 2)$, *are invariants of the group* G_{gas}.

Proposition 4.1. *Between the invariants in Theorem* 4.2 *there exist the following relations:*

$$\begin{cases} \mathcal{S}_{ijk}\left\{\underset{(11)}{\overset{1}{k}}_{ijk}\right\} = 2\underset{(0)}{T}{}^*_{ijk} + \underset{(1)}{t}{}^*_{ijk} , \quad \mathcal{S}_{ijk}\left\{\underset{(\alpha)}{\overset{2}{k}}_{ijk}\right\} = 2\underset{(\alpha)}{S}{}^*_{ijk} , \\ \mathcal{S}_{ijk}\left\{\underset{(11)}{\overset{3}{k}}_{ijk}\right\} = \underset{(0)}{T}{}^*_{ijk} + \underset{(1)}{t}{}^*_{ijk} , \quad \mathcal{S}_{ijk}\left\{\underset{(\alpha)}{\overset{4}{k}}_{ijk}\right\} = \underset{(\alpha)}{S}{}^*_{ijk} , \\ \underset{(\alpha)}{\overset{2}{k}}_{ijk} + \underset{(\alpha)}{\overset{4}{k}}_{ijk} = \underset{(\alpha)}{S}{}^*_{ijk} , \quad (\alpha = 1, 2). \end{cases} \tag{4.5}$$

References

1. Gh. Atanasiu, The equations of structure of an *N*-linear connection in the bundle of accelerations, *Balkan Journal of Geometry and Its Applications*, **1** (1996), 11−19.
2. Gh. Atanasiu and I. Ghinea, Connexions finsleriennes générales presque symplectiques, *An. Sti. Univ. "Al.I. Cuza" Iasi, Sect. I a Mat.* **25** (Supl.) (1979), 11−15.
3. V. Cruceanu and R. Miron, Sur les connexions compatible à une structure mêtrique ou presque symplectique, *Mathematica (Cluj)* **9** (1967), 245−252.
4. M. Matsumoto, *The Theory of Finsler Connections*, Publ. Study Group Geom. 5, Depart. Math., Okayama Univ., 1970.
5. R. Miron and Gh. Atanasiu, Lagrange geometry of second order, *Math. Comput. Modelling, Pergamon* **20** (1994), 41−56.
6. R. Miron and Gh. Atanasiu, Differential geometry of the *k*-osculator bundle, *Rev. Roumaine Math. Pures Appl.* **41** (1996), 205−236.
7. R. Miron and Gh. Atanasiu, Prolongation of Riemannian, Finslerian and Lagrangian structures, *Rev. Roumaine Math. Pures Appl.* **41** (1996), 237−249.
8. R. Miron and Gh. Atanasiu, Higher-order Lagrange spaces, *Rev. Roumaine Math. Pures Appl.* **41** (1996), 251−262.
9. R. Miron and Gh. Atanasiu, *Compendium sur les espaces Lagrange d'ordre supérieur*, Seminarul de Mecanica 40, Univ. Timisoara, 1994.
10. R. Miron and M. Hashiguchi, Metrical Finsler connections, *Fac. Sci., Kagoshima Univ. (Math., Phys. & Chem.)* **12** (1979), 21−35.
11. R. Miron and M. Hashiguchi, Almost symplectic Finsler structures, *Fac. Sci., Kagoshima Univ. (Math., Phys. & Chem.)* **14** (1981), 9−19.
12. R. Miron, *The Geometry of Higher-order Lagrange Spaces.* Applications in Mechanics and Physics, Kluwer Academic Publishers (FTPH 82, 1997).
13. R. Miron, On transformation groups of Finsler connections, *Tensor, N.S* **35** (1981), 235−240.
14. E. Stoica and M. Purcaru, General *N*-linear connections compatible with the almost symplectic *d*-structures in the bundle of accelerations, *Tensor, N.S.*, (to appear).
15. M. Purcaru, I. Sofroniciu and M. Paun, The almost symplectic structures in the bundle space of the accelerations, *Proceedings of the 23rd Conference on Geometry and Topology*, (1993), 141−146.

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