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The Hahn Sequence Space-III

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Abstract. The purpose of this article is to introduce a new class of sequence spaces, namely semi-Hahn spaces. It is shown that the intersection of all semi-Hahn spaces is the Hahn space. There are already semi-conservative and semi-replete spaces in literature. However, there is no relation among these spaces. Some properties of the Hahn space which are not included in [2] and [3] are given, before introducing semi-Hahn spaces. In fact, conditions for inclusion of the Hahn space h in an FK-space, AB-property of h, and inclusion of the distinguished subspace $B^+(h)$ are studied.

1. Introduction

This paper is a continuation of the papers by K. Chandrasekhara Rao [2] and by K. Chandrasekhara Rao and T.G. Srinivasalu [3].

We use the following notation:
A (complex) sequence whose k^{th} term is x_k will be denoted by (x_k) or x.
Let

Φ	=	the set of all finite sequences.
ℓ_∞	=	the Banach space of all bounded sequences .
w	=	all complex sequences.
cs	=	the Banach space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} x_k$ converges.
bs	=	the Banach space of all sequences $x = (x_k)$ such that $\sup_{(n)} \left\| \sum_{k=1}^{n} x_k \right\|$ exists .

bv = the Banach space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} |x_k - x_{k+1}|$ converges.

$\sigma(\ell_\infty)$ = the Cesàro space of ℓ_∞

$$= \left\{ x = (x_k) : (y_k) \in \ell_\infty, y_k = \frac{x_1 + x_2 + \cdots + x_k}{k} \text{ for } k = 1, 2, 3, \cdots \right\}.$$

h = The Hahn space is the BK space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} k|x_k - x_{k+1}|$ converges and $\lim_{k\to\infty} x_k = 0$.

The norm on h is given by $$\|x\| = \sum_{k=1}^{\infty} k|x_k - x_{k+1}| = \sum_{k=1}^{\infty} k|\Delta x_k|$$

where $\Delta x_k = (x_k - x_{k+1})$, $k = 1, 2, 3, \cdots$.

Given a sequence $x = (x_k)$ its nth section is the sequence $x^{(n)} = \{x_1, \cdots, x_n, 0, 0, \cdots\}$.

$$\delta^{(k)} = \{0, 0, \cdots, 1, 0, 0, \cdots\} \text{ for } k = 1, 2, 3, \cdots.$$

k^{th} place

We recall the following definitions (see [1]).

An FK-space is a locally convex Fréchet space which is made up of sequences and has the property that coordinate projections are continuous.

A BK-space is a locally convex Banach space which is made up of sequences and has the property that coordinate projections are continuous.

A BK-space X is said to have AK (or Sectional convergence) if and only if $\| x^{(n)} - x \| \to 0$ as $n \to \infty$ [5].

Let X be an FK-space. A sequence (x_k) in X is said to be weakly Cesàro bounded if $\left\{ \frac{f(x_1) + f(x_2) + \cdots + f(x_k)}{k} \right\}$ is bounded for each $f \in X'$, the dual space of X.

The space X is said to have AD (or) be an AD space if Φ is dense in X.

We note that

AK $\Rightarrow$ AD by [8].

An FK-space $X \supset \Phi$ is said to have AB if $(x^{(n)})$ is a bounded set in X for each $x \in X$.

Let X be a BK-space. Then X is said to have monotone norm if $\| x^{(m)} \| \geq \| x^{(n)} \|$ for $m > n$ and $\| x \| = \sup_{(n)} \| x^{(n)} \|$.

Let X be an FK-space $\supset \Phi$. Then

$$B^{+} = X^{f \gamma} = B^{+}(X) = \{z \in w : (z^{(n)}) \text{ is bounded in } X\}$$

$$= \{z \in w : (z_n f(\delta^{(n)}) \in bs \quad \forall \quad f \in X'\}$$

Also we write

$$B = B^{+} \cap X.$$

Let X be a BK-space. Then X is an AB-space if and only if $B = X$. Any space with monotone norm has AB (see Theorem 10.3.12 of [1])

If X is a sequence space, we define

(i) $X' =$ the continuous dual of X.

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\};$

(iii) $X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\};$

(iv) $X^{\gamma} = \{a = (a_k) : \sup_{(n)} \left| \sum_{k=1}^{n} a_k x_k \right| < \infty, \text{ for each } x \in X\}.$

(v) Let X be an FK-space $\supset \Phi$. Then $X^{f} = \{(f(\delta^{(n)}) : f \in X'\}.$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the

- α - (or Köthe-Toeplitz), dual of X.
- β - (or generalized Köthe-Toeplitz), dual of X.
- γ - dual of X.

X^{f} is called the f-dual of X. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$.

If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta,$ or γ.

Lemma 1. *(See Theorem 7.2.7. in [1]). Let X be an FK-space $\supset \Phi$. Then*

(i) $X^{\beta} \subset X^{\gamma} \subset X^{f}$.

(ii) *If X has AK,* $X^{\beta} = X^{f}$.

(iii) *If X has AD,* $X^{\beta} = X^{\gamma}$.

The following facts reveal the importance of the Hahn sequence space:

1. The Hahn sequence is the smallest semi replete space (see [3]).
2. The Hahn sequence space is an example of a Banach space which is not rotund (see Proposition-2 of [2]).
3. $bv_0 \cap d\ell = dh$

   where $d\ell = \{x = (x_k) : \sum_{k=1}^{\infty} \frac{|x_k|}{k} < \infty\}$.

$$bv_0 = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty \text{ and } \lim_{k\to\infty} = 0\}\ x_k$$

$$dh = \{x = (x_k) : \sum_{k=1}^{\infty} k\left|\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\}$$

   (See Theorem 3.2 of [5]).

As pointed out by Wilansky in [1], because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi conservative.

Snyder and Wilansky introduced the concept of semi conservative spaces in [6]. Snyder studied the properties of semi conservative spaces in [7]. Later on, in the year 1996 K. Chandrasekhara Rao and T.G. Srinivasalu introduced the semi replete spaces in [3].

Given the Hahn sequence space h it is our aim to find a sequence space X such that h is the intersection of all such X. Such a space X is called a semi-Hahn space.

Thus semi-Hahn spaces are introduced and their properties are studied.

As a prelude, some properties of the Hahn sequence space are also given for the sake of completeness of the paper. These properties did not appear in [2] and [3].

2. Results

Lemma 2. $h^f = \sigma(\ell_\infty)$.

Proof: $h^\beta = \sigma(\ell_\infty)$ by Theorem 3.1 in [5]. But *h* has AK. (see Goes and Goes [5]). Hence $h^\beta = h^f$. Therefore $h^f = \sigma(\ell_\infty)$. This completes the proof.

Theorem 1. *Let Y be any FK-space* $\supset \Phi$. *Then* $Y \supset h$ *if and only if the sequence* $\{\delta^{(k)}\}$ *is weakly Cesàro bounded.*

Proof. We know that h has AK. But every AK-space is AD (see [8]). Now the following implications establish the result.

$Y \supset h \iff Y^f \subset h^f$ since h has AD and hence by using Theorem 8.6.1 of [1].

$\iff Y^f \subset \sigma(\ell_\infty)$ by Lemma 2.

$\iff$ for each $f \in Y'$, the topological dual of Y, $f(\delta^{(k)}) \in \sigma(\ell_\infty)$.

$$\iff \left\{ \frac{f(\delta^{(1)}) + f(\delta^{(2)}) + f(\delta^{(3)}) + \cdots + f(\delta^{(k)})}{k} \right\} \in \ell_\infty .$$

$$\iff \left\{ \frac{f(\delta^{(1)}) + f(\delta^{(2)}) + f(\delta^{(3)}) + \cdots + f(\delta^{(k)})}{k} \right\} \text{ is bounded.}$$

$\iff$ The sequence $\{\delta^{(k)}\}$ is weakly Cesàro bounded. This completes the proof.

Theorem 2. *Suppose that* h *is a closed subspace of an FK-space X. Then* $B^+(X) \subset h$.

Proof. Note that c_0 has AK. Hence $\sigma(c_0)$ has AK. Consequently $\sigma(c_0)$ has AD. Therefore by Lemma 1, $[\sigma(c_0)]^\beta = [\sigma(c_0)]^\gamma$. By Theorem 10.3.5 of [1] we have

$$B^+(X) = B^+(h)$$
$$= h^{f\gamma}$$
$$= (h^f)^\gamma = \{\sigma(\ell_\infty)\}^\gamma,$$

by Lemma 2.But $(\sigma(\ell_\infty))^\gamma \subset (\sigma(c_0))^\gamma = (\sigma(c_0))^\beta$ and $(\sigma(c_0))^\beta = h$. (see page 97 of [5]). Hence $B^+(X) \subset h$. This completes the proof.

Theorem 3. *Let X be an AK-space* $\supset \Phi$. *Then* $X \supset h$ *if and only if* $B^{+}(X) \supset h$.

Proof. (Necessity). Suppose that $X \supset h$. Then

$$B^{+}(X) \supset B^{+}(h) \tag{3.1}$$

by the monotonicity Theorem 10.2.9 of [1]. By Theorem 10.3.4 of [1] we have

$$B^{+}(h) = h^{f\gamma} = h \tag{3.2}$$

From (3.1) and (3.2) we obtain $B^{+}(X) \supset B^{+}(h) = h$.
(Sufficiency). Suppose that $B^{+}(X) \supset h$. We have

$$h^{\gamma} \supset [B^{+}(X)]^{\gamma} \tag{3.3}$$

But h has AK and so h has AD. Therefore

$$h^{\beta} = h^{\gamma} = h^{f} \tag{3.4}$$

But always

$$B^{+}(X) = X^{f\gamma} \tag{3.5}$$

From (3.3) and (3.5) $h^{\gamma} \supset (X^{f\gamma})^{\gamma} = (X^{f})^{\gamma\gamma} \supset X^{f}$. Thus from (3.4) $h^{f} \supset X^{f}$. Now by Theorem 8.6.1 of [1]; since h has AD we conclude that $X \supset h$. This completes the proof.

Theorem 4. *The space h has monotone norm.*

Proof. Let $m > n$. It follows from

$$|x_n| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots + |x_{m-1} - x_m| + |x_m|$$

that

$$\left\|x^{(n)}\right\| \le \left(\sum_{k=1}^{n-1} k|x_k - x_{k+1}|\right) + n|x_n - x_{n+1}| + \ldots + (m-1)|x_{m-1} - x_m| + m|x_m| = \left\|x^{(m)}\right\|$$

The sequence $(\|x^{(x)}\|)$ being monotone increasing, it thus follows from $x = \lim_{n\to\infty} x^{(n)}$ that

$$\|x\| = \lim_{n\to\infty}\left\|x^{(n)}\right\| = \sup_{(n)}\left\|x^{(n)}\right\|.$$

This completes the proof.

Definition. *An FK-space X is called "semi-Hahn" if* $X^f \subset \sigma(\ell_\infty)$. *In other words*

$$f(\delta^{(k)}) \in \sigma(\ell_\infty) \ \forall \ f \in X'.$$

$$\Leftrightarrow \left\{\frac{f(\delta^{(1)}) + f(\delta^{(2)}) + \cdots + f(\delta^{(k)})}{k}\right\} \in \ell_\infty$$

$$\Leftrightarrow \left\{\frac{f(\delta^{(1)}) + f(\delta^{(2)}) + \cdots + f(\delta^{(k)})}{k}\right\} \text{ is bounded for each } f \in X'.$$

Example. The Hahn space is semi-Hahn. Indeed, if h be Hahn space, then by Lemma 2, $h^f = \sigma(\ell_\infty)$.

We recall

Lemma 3. (4.3.7 *of* [1]). *Let* z *be a sequence. Then* (z^β, p) *is an AK space with* $p = (p_k : k = 0, 1, 2, \cdots)$ *where*

$$p_0(x) = \sup_m \left|\sum_{k=1}^m z_k x_k\right|, \quad p_n(x) = |x_n|.$$

For any k such that $z_k \neq 0$, p_k *may be omitted. If* $z \in \Phi$, p_0 *may be omitted.*

Theorem 5. z^β *is semi-Hahn if and only if* $z \in \sigma(\ell_\infty)$.

Proof.

Step 1. Suppose that z^β is semi-Hahn. z^β has AK by Lemma 3. Hence $z^\beta = z^f$.

Therefore $z^{\beta\beta} = (z^\beta)^f$ by Theorem 7.2.7 of [1]. So z^β is semi-Hahn if and only if $z^{\beta\beta} \subset \sigma(\ell_\infty)$. But then $z \in z^{\beta\beta} \subset \sigma(\ell_\infty)$.

Step 2. Conversely, let $z \in \sigma(\ell_\infty)$. Then $z^\beta \supset \{\sigma(\ell_\infty)\}^\beta$ and $z^{\beta\beta} \subset \{\sigma(\ell_\infty)\}^{\beta\beta}$ $= h^\beta = \sigma(\ell_\infty)$. But $(z^\beta)^f = z^{\beta\beta}$. Hence $(z^\beta)^f \subset \sigma(\ell_\infty) \Rightarrow z^\beta$ is semi-Hahn. This completes the proof.

Theorem 6. *Every semi-Hahn space contains h.*

Proof. Let X be any semi-Hahn space.

$$\Rightarrow X^f \subset \sigma(\ell_\infty).$$

$$\Rightarrow f(\delta^{(k)}) \in \sigma(\ell_\infty) \ \forall \ f \in X'.$$

$\Rightarrow \left\{\delta^{(k)}\right\}$ is weakly Cesàro bounded w.r. to X.

$\Rightarrow X \supset h$ by Theorem 1.

This completes the proof.

Theorem 7. *The intersection of all semi-Hahn spaces is h.*

Proof. Let I be the intersection of all semi-Hahn spaces. Then the intersection

$$\begin{aligned} I &\subset \bigcap \left\{z^\beta : z \in \sigma(\ell_\infty)\right\} \\ &= \left\{\sigma(\ell_\infty)\right\}^\beta \\ &= h \end{aligned} \tag{7.1}$$

By Theorem 6,

$$h \subset I \tag{7.2}$$

From (7.1) and (7.2) we get

$$I = h.$$

This completes the proof.

Corollary. *The smallest semi-Hahn space is h.*

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