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On δD-Sets and Associated Weak Separation Axioms

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Abstract. Veličko [4] introduced the notions of δ-open sets and δ-closure. In this paper, we introduce some weak separation axioms by utilizing δ-open sets and the δ-closure operator.

1. Introduction

Throughout this paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces. A subset A of a topological space X is said to be *regular open* (resp. *regular closed*) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$), where $Int(A)$ and $Cl(A)$ the interior and the closure of a set A. A point $x \in X$ is called the δ-*cluster point* of A if $A \cap U \neq \varnothing$ for every regular open set U of X containing x. The set of all δ-cluster points of A is called the δ-*closure* of A, denoted by $Cl_\delta(A)$. A subset A is called δ-*closed* if $A = Cl_\delta(A)$. The complement of a δ-closed sets is called δ-*open*. We denote the collection of all δ-open (resp. δ-closed) sets by $\delta O(X, \tau)$ (resp. $\delta C(X, \tau)$). A set U is a δ-*neighborhood* of a point x if U is δ-open such that $x \in U$.

Lemma 1.1. *Intersection of arbitrary of δ-closed sets in (X, τ) is δ-closed.*

In what follows, (X, τ) is a regular topological space.

Corollary 1.2. *Let A be a subset of a topological space (X, τ),* $Cl_\delta(A) = \cap\{F \in \delta C(X, \tau) \mid A \subset F\}$.

Corollary 1.3. $Cl_\delta(A)$ *is δ-closed, that is* $Cl_\delta(Cl_\delta(A)) = Cl_\delta(A)$.

Lemma 1.4. *For subsets A and $A_i (i \in I)$ of a space (X, τ), the following hold:*

(1) $A \subset Cl_\delta(A)$.

(2) If $A \subset B$, *then* $Cl_\delta(A) \subset Cl_\delta(B)$.

(3) $Cl_\delta(\bigcap\{A_i : i \in I\}) \subset \bigcap\{Cl_\delta(A_i) : i \in I\}$.

(4) $Cl_\delta(\bigcup\{A_i : i \in I\}) = \bigcup\{Cl_\delta(A_i) : i \in I\}$.

2. *δD*-sets and associated separation axioms

Definition 1. *A subset A of a topological space X is called a δD-sets if there are two U, V $\in \delta O(X, \tau)$ such that $U \neq X$ and $A = U - V$.*

Clearly every δ-open set *U* different from *X* is a δD-set if $A = U$ and $V = \varnothing$.

Definition 2. *A topological space (X, τ) is called δ-D_0 if for any distinct pair of points x and y of X there exists a δD-set of X containing x but not y or a δD-set of X containing y but not x.*

Definition 3. *A topological space (X, τ) is called δ-D_1 if for any distinct pair of points x and y of X there exists a δD-set of X containing x but not y and a δD-set of X containing y but not x.*

Definition 4. *A topological space (X, τ) is called δ-D_2 if for any distinct pair of points x and y of X there exists disjoint δD-sets G and E of X containing x and y, respectively.*

Definition 5. *A topological space (X, τ) is called δ-T_0* [2] *if for any distinct pair of points in X, there is a δ-open set containing one of the points but not the other.*

Definition 6. *A topological space (X, τ) is called δ-T_1* [2] *if for any distinct pair of points x and y in X, there is a δ-open U in X containing x but not y and a δ-open set V in X containing y but not x.*

Definition 7. *A topological space (X, τ) is called δ-T_2* [2] *if for any distinct pair of points x and y in X, there exist δ-open sets U and V in X containing x and y, respectively, such that $U \cap V = \varnothing$.*

Remark 2.1. (i) If (X, τ) is δ-T_i, then it is δ-T_{i-1}, $i = 1, 2$.

(ii) Obviously, if (X, τ) is δ-T_i, then (X, τ) is δ-D_i, $i = 0, 1, 2$.

(iii) If (X, τ) is δ-D_i, then it is δ-D_{i-1}, $i = 1, 2$.

Theorem 2.2. *For a topological space* (X, τ) *the following statements are true:*

(1) (X, τ) *is* δ-D_0 *if and only if it is* δ-T_0.

(2) (X, τ) *is* δ-D_1 *if and only if it is* δ-D_2.

Proof. (1) The sufficiency is stated in Remark 2.1(ii). To prove necessity, let (X, τ) be δ-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y, say x, belongs to a δD-set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and $U_1, U_2 \in \delta O(X, \tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$;

In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is δ-T_0.

(2) Sufficiency. Remark 2.1(iii).

Necessity. Suppose X δ-D_1. Then for each distinct pair $x, y \in X$, we have δD-sets G_1, G_2 such that $x \in G_1$; $y \notin G_1$; $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2, G_2 = U_3 \setminus U_4$. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$ and by $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4) = \varnothing$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, $y \in U_2$. $(U_1 \setminus U_2) \cap U_2 = \varnothing$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4, x \in U_4$. $(U_3 \setminus U_4) \cap U_4 = \varnothing$. Therefore X is δ-D_2.

Corollary 2.3. *If* (X, τ) *is* δ-D_1, *then it is* δ-T_0.

Theorem 2.4. *A topological space* (X, τ) *is* δ-T_0 *if and only if for each pair of distinct points x, y of X,* $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

Proof. Sufficiency: Suppose that $x, y \in X$, $x \neq y$ and $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Let z be a point of X such that $z \in Cl_\delta(\{x\})$ but $z \notin Cl_\delta(\{y\})$. We claim that $x \notin Cl_\delta(\{y\})$. For, if $x \in Cl_\delta(\{y\})$ then $Cl_\delta(\{x\}) \subset Cl_\delta(\{y\})$. This contradicts the fact that $z \notin Cl_\delta(\{y\})$. Consequently x belongs to the δ-open set $[Cl_\delta(\{y\})]^c$ to which y does not belong.

Necessity: Let (X,τ) be a δ-T_0 space and *x, y* be any two distinct points of *X*. There exists a δ-open set *G* containing *x* or *y*, say *x* but not *y*. Then G^c is a δ-closed set which does not contain *x* but contains *y*. Since $Cl_\delta(\{y\})$ is the smallest δ-closed set containing *y* (Corollary 1.2), $Cl_\delta(\{y\}) \subset G^c$, and therefore $x \notin Cl_\delta(\{y\})$. Consequently $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

Theorem 2.5. *A topological space* (X,τ) *is* δ-T_1 *if and only if the singletons are* δ*-closed sets.*

Proof. Let (X,τ) be δ-T_1 and *x* any point of *X*. Suppose $y \in \{x\}^c$. Then $x \neq y$ and so there exists a δ-open set U_y such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup\{U_y \,/\, y \in \{x\}^c\}$ which is δ-open.

Conversely. Suppose $\{p\}$ is δ-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is a δ-open set containing *y* but not *x*. Similarly $\{y\}^c$ is a δ-open set containing *x* but not *y*. Accordingly *X* is a δ-T_1 space.

Definition 8. *A point* $x \in X$ *which has X as the unique* δ*-neighborhood is called* δ*-neat point.*

Theorem 2.6. *For a* δ-T_0 *topological space* (X,τ) *the following are equivalent:*

(1) (X,τ) *is* δ-D_1 *;*

(2) (X,τ) *has no* δ*-neat point.*

Proof. (1) $\rightarrow$ (2). Since (X,τ) is δ-D_1, then each point *x* of *X* is contained in a δD-set $O = U - V$ and thus in *U*. By definition $U \neq X$. This implies that *x* is not a δ-neat point.

(2) $\rightarrow$ (1). If *X* is δ-T_0, then for each distinct pair of points $x, y \in X$, at least one of them, *x* (say) has a δ-neighborhood *U* containing *x* and not *y*. Thus *U* which is different from *X* is a δD-set. If *X* has no δ-neat point, then *y* is not a δ-neat point. This means that there exists a δ-neighborhood *V* of *y* such that $V \neq X$. Thus $y \in (V - U)$ but not *x* and $V - U$ is a δD-set. Hence *X* is δ-D_1.

Remark 2.7. It is clear that a δ-T_0 topological space (X,τ) is not δ-D_1 if and only if there is a unique δ-neat point in *X*. It is unique because if *x* and *y* are both δ-neat point in *X*, then at least one of them say *x* has a δ-neighborhood *U* containing *x* but not *y*. But this is a contradiction since $U \neq X$.

Definition 9. *A topological space* (X,τ) is *δ-symmetric if for x and y in X,* $x \in Cl_\delta(\{y\})$ implies $y \in Cl_\delta(\{x\})$.

Definition 10. *A subset A of a topological space* (X,τ) *is called a* (δ,δ)*-generalized-closed set* [1] *(briefly* (δ,δ)*-g-closed) if* $Cl_\delta(A) \subset U$ *whenever* $A \subset U$ *and U is δ-open in* (X,τ)*.*

Lemma 2.8. *Every δ-closed set is* (δ,δ)*-g-closed.*

Theorem 2.9. *A topological space* (X,τ) *is δ-symmetric if and only if* $\{x\}$ *is* (δ,δ)*-g-closed for each* $x \in X$*.*

Proof. Assume that $x \in Cl_\delta(\{y\})$ but $y \notin Cl_\delta(\{x\})$. This means that $[Cl_\delta(\{x\})]^c$ contains *y*. This implies that $Cl_\delta(\{y\})$ is a subset of $[Cl_\delta(\{x\})]^c$. Now $[Cl_\delta(\{x\})]^c$ contains *x* which is a contradiction.

Conversely, suppose that $\{x\} \subset E \in \delta(X,\tau)$ but $Cl_\delta(\{x\})$ is not a subset of *E*. This means that $Cl_\delta(\{x\})$ and E^c are not disjoint. Let *y* belongs to their intersection. Now we have $x \in Cl_\delta(\{y\})$ which is a subset of E^c and $x \notin E$. But this is a contradiction.

Corollary 2.10. *If a topological space* (X,τ) *is a* δ-T_1 *space, then it is δ-symmetric.*

Proof. In a δ-T_1 space, singleton sets are δ-closed (Theorem 2.5) and therefore (δ,δ)*-g-closed* (Lemma 2.8). By Theorem 2.9, the space is δ-symmetric.

Corollary 2.11. *For a topological space* (X,τ) *the following are equivalent:*

(1) (X,τ) *is δ-symmetric and* δ-T_0*;*

(2) (X,τ) *is* δ-T_1.

Proof. By Corollary 2.10 and Remark 2.1 it suffices to prove only (1) $\rightarrow$ (2). Let $x \neq y$ and by δ-T_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \delta O(X,\tau)$. Then $x \notin Cl_\delta(\{y\})$ and hence $y \notin Cl_\delta(\{x\})$. There exists a $G_2 \in \delta O(X,\tau)$ such that $y \in G_2 \subset \{x\}^c$ and (X,τ) *is a* δ-T_1 space.

Theorem 2.12. *For a δ-symmetric topological space (X,τ) the following are equivalent:*

(1) (X,τ) *is* δ-T_0;

(2) (X,τ) *is* δ-D_1;

(3) (X,τ) *is* δ-T_1.

Proof. $(1) \to (3)$: Corollary 2.11.
$(3) \to (2) \to (1)$: Remark 2.1.

Definition 11. *A function* $f : (X,\tau) \to (Y,\sigma)$ *is said to be* δ*-continuous* [3] *if for each* $x \in X$ *and each regular open set V containing* $f(x)$, *there is a regular open set U in X containing x such that* $f(U) \subset V$.

Remark 2.13. In 1980, Noiri [3] proved that a function $f : (X,\tau) \to (Y,\sigma)$ is δ-continuous if and only if the inverse image of each δ-open set is δ-open.

Theorem 2.14. *If* $f : (X,\tau) \to (Y,\sigma)$ *is a* δ*-continuous surjective function and E is a* δD*-set in Y, then the inverse image of E is a* δD*-set in X.*

Proof. Let E be a δD-set in Y. Then there are δ-open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. By the δ-continuity of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are δ-open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)$ is a δD-set.

Theorem 2.15. *If* (Y,σ) *is* δ-D_1 *and* $f : (X,\tau) \to (Y,\sigma)$ *is* δ*-continuous and bijective, then* (X,τ) *is* δ-D_1.

Proof. Suppose that Y is a δ-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is δ-D_1, there exist δD-sets G_z and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_z$ and $f(x) \notin G_y$. By Theorem 2.14, $f^{-1}(G_z)$ and $f^{-1}(G_y)$ are δD-sets in X containing x and y, respectively. This implies that X is a δ-D_1 space.

Theorem 2.16. *A topological space* (X,τ) *is* δ-D_1 *if and only if for each pair of distinct points* $x, y \in X$, *there exists a* δ*-continuous surjective function* $f : (X,\tau) \to (Y,\sigma)$, *where Y is a* δ-D_1 *space such that* $f(x)$ *and* $f(y)$ *are distinct.*

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.
Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a δ-continuous, surjective function f of a space X onto a δ-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint δD-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is δ-continuous and surjective, by Theorem 2.14, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint δD-sets in X containing x and y, respectively. Hence by Theorem 2.2, X is δ-D_1 space.

3. Sober δ-R_0 spaces

Definition 12. *Let A be a subset of topological space X. The δ-kernel of A, denoted by $Ker_\delta(A)$ is defined to be the set $Ker_\delta(A) = \cap\{O \in \delta O(X,\tau) \mid A \subset O\}$.*

Lemma 3.1. *Let (X,τ) be a topological space and $x \in X$. Then $Ker_\delta(A) = \{x \in X \,/\, Cl_\delta(\{x\}) \cap A \neq \varnothing\}$.*

Proof. Let $x \in Ker_\delta(A)$ and suppose $Cl_\delta(\{x\}) \cap A = \varnothing$. Hence $x \notin [Cl_\delta(\{x\})]^c$ which is a δ-open set containing A. This is absurd, since $x \in Ker_\delta(A)$. Consequently, $Cl_\delta(\{x\}) \cap A \neq \varnothing$. Next, let $Cl_\delta(\{x\}) \cap A \neq \varnothing$ and suppose that $x \notin Ker_\delta(A)$. Then, there exists a δ-open set D containing A and $x \notin D$. Let $y \in Cl_\delta(\{x\}) \cap A$. Hence, D is a δ-neighborhood of y which $x \notin D$. By this contradiction $x \in Ker_\delta(A)$ and the claim.

Definition 13. *A topological space (X,τ) is said to be sober δ-R_0 if $\cap_{x\in X} Cl_\delta(\{x\}) = \varnothing$.*

Therefore 3.2. *A topological space (X,τ) is sober δ-R_0 if and only if $Ker_\delta(\{x\}) \neq X$ for every $x \in X$.*

Proof. Suppose that the space (X,τ) *be sober* δ-R_0. Assume that there is a point y in X such that $Ker_\delta(\{y\}) = X$. Then $y \notin O$ which O is some proper δ-open subset of X. This implies that $y \in \cap_{x\in X} Cl_\delta(\{x\})$. But this is a contradiction.

Now assume that $Ker_\delta(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap_{x\in X} Cl_\delta(\{x\})$, then every δ-open set containing y must contain every point of X. This implies that the space X is the unique δ-open set containing y. Hence $Ker_\delta(\{y\}) = X$ which is a contradiction. Therefore (X,τ) is sober δ-R_0.

Definition 14. *A function* $f : X \to Y$ *is called always* δ*-closed if the image of every* δ*-closed subset of X is* δ*-closed in Y.*

Theorem 3.3. *If* $f : X \to Y$ *is an bijective always* δ*-closed function and X is sober* δ-R_0, *then Y is sober* δ-R_0.

Proof. Straightforward.

Theorem 3.4. *If the topological space X is sober* δ-R_0 *and Y is any topological space, then the product* $X \times Y$ *is sober* δ-R_0.

Proof. By showing that $\cap_{(x,y)\in X\times Y} Cl_\delta(\{x, y\}) = \varnothing$ we are done. We have:

$$\begin{aligned}\cap_{(x,y)\in X\times Y} Cl_\delta(\{x, y\}) &\subseteq \cap_{(x,y)\in X\times Y} (Cl_\delta(\{x\}) \times Cl_\delta(\{y\})) \\ &= \cap_{x\in X} Cl_\delta(\{x\}) \times \cap_{y\in Y} Cl_\delta(\{y\}) \subseteq \varnothing \times Y = \varnothing.\end{aligned}$$

4. δ-R_0 spaces and δ-R_1 spaces

Definition 15. *A topological space* (X, τ) *is said to be* δ-R_0 *space* [1] *if every* δ*-open set contains the* δ*-closure of each of its singletons.*

Definition 16. *A topological space* (X, τ) *is said to be* δ-R_1 *if for* x, y *in* X *with* $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$, *there exist disjoint* δ*-open sets U and V such that* $Cl_\delta(\{x\})$ *is a subset of U and* $Cl_\delta(\{y\})$ *is a subset of V.*

Lemma 4.1. *Let* (X, τ) *be a topological space and* $x \in X$. *Then* $y \in Ker_\delta(\{x\})$ *if and only if* $x \in Cl_\delta(\{y\})$.

Proof. Suppose that $y \notin Ker_\delta(\{x\})$. Then there exists a δ-open set *V* containing *x* such that $y \notin V$. Therefore we have $x \notin Cl_\delta(\{y\})$. The proof of converse case can be done similarly.

Lemma 4.2. *The following statements are equivalent for any points x and y in a topological space* (X, τ) *:*

(1) $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$;

(2) $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

Proof. (1) $\rightarrow$ (2) : Suppose that $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$, then there exists a point z in X such that $z \in Ker_{\delta}(\{x\})$ and $z \notin Ker_{\delta}(\{y\})$. From $z \in Ker_{\delta}(\{x\})$ it follows that $\{x\} \cap Cl_{\delta}(\{z\}) \neq \varnothing$ which implies $x \in Cl_{\delta}(\{z\})$. By $z \notin Ker_{\delta}(\{y\})$, we have $\{y\} \cap Cl_{\delta}(\{z\}) = \varnothing$. Since $x \in Cl_{\delta}(\{z\})$, $Cl_{\delta}(\{x\}) \subset Cl_{\delta}(\{z\})$ and $\{y\} \cap Cl_{\delta}(\{x\}) = \varnothing$. Therefore it follows that $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$. Now $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$ implies that $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$.
(2) $\rightarrow$ (1) : Suppose that $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$. Then there exists a point z in X such that $z \in Cl_{\delta}(\{x\})$ and $x \notin Cl_{\delta}(\{y\})$. Then there exists a δ-open set containing z and therefore x but not y, namely, $y \notin Ker_{\delta}(\{x\})$. Hence $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$.

Theorem 4.3. *If (X, τ) is δ-R_1, then (X, τ) is δ-R_0.*

Proof. Let U be δ-open and $x \in U$. If $y \notin U$, then since $x \notin Cl_{\delta}(\{y\})$, $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$. Hence, there exists a δ-open V_y such that $Cl_{\delta}(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin Cl_{\delta}(\{x\})$. Thus $Cl_{\delta}(\{x\}) \subset U$. Therefore (X, τ) is δ-R_0.

Theorem 4.4. *A topological space (X, τ) is δ-R_1 if and only if for $x, y \in X$, $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$, there exist disjoint δ-open sets U and V such that $Cl_{\delta}(\{x\}) \subset U$ and $Cl_{\delta}(\{y\}) \subset V$.*

Proof. It follows from Lemma 4.2.

Theorem 4.5. *A topological space (X, τ) is a δ-R_0 space if and only if for any x and y in X, $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$ implies $Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\}) = \varnothing$.*

Proof. Suppose that (X, τ) is δ-R_0 and $x, y \in X$ such that $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$. Then, there exist $z \in Cl_{\delta}(\{x\})$ such that $z \notin Cl_{\delta}(\{y\})$ (or $z \in Cl_{\delta}(\{y\})$) such that $z \notin Cl_{\delta}(\{x\})$. There exists $V \in \delta O(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin Cl_{\delta}(\{y\})$. Thus $x \in [Cl_{\delta}(\{y\})]^c \in \delta O(X, \tau)$, which implies $Cl_{\delta}(\{x\}) \subset [Cl_{\delta}(\{y\})]^c$ and $Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\}) = \varnothing$. the proof for otherwise is similar.

Sufficiency: Let $V \in \delta O(X, \tau)$ and let $x \in V$. We still show that $Cl_{\delta}(\{x\}) \subset V$. Let $y \notin V$, i.e., $y \in [V]^c$. Then $x \neq y$ and $x \notin Cl_{\delta}(\{y\})$. This shows that $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$. By assumption, $Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\}) = \varnothing$. Hence $y \notin Cl_{\delta}(\{x\})$. Therefore $Cl_{\delta}(\{x\}) \subset V$.

Theorem 4.6. *A topological space (X,τ) is a δ-R_0 space if and only if for any points x and y in X, $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ implies $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \varnothing$.*

Proof. Suppose that (X,τ) is a δ-R_0 space. Thus by Lemma 4.2. for any points x and y in X if $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ then $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Now we prove that $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \varnothing$. Assume that $z \in Ker_\delta(\{x\}) \cap Ker_\delta(\{y\})$. By $z \in Ker_\delta(\{x\})$ and Lemma 4.1, it follows that $x \in Cl_\delta(\{z\})$. Since $x \in Cl_\delta(\{x\})$, by Theorem 4.5 $Cl_\delta(\{x\}) = Cl_\delta(\{z\})$. Similarly, we have $Cl_\delta(\{y\}) = Cl_\delta(\{z\}) = Cl_\delta(\{x\})$. This is a contradiction. Therefore, we have $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \varnothing$.

Conversely, let (X,τ) be a topological space such that for any points x and y in X, $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$, then by Lemma 4.2, $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$. Hence by hypothesis $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \varnothing$ which implies $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \varnothing$. Because $z \in Cl_\delta(\{x\})$ implies that $x \in Ker_\delta(\{z\})$ and therefore $Ker_\delta(\{x\}) \cap Ker_\delta(\{z\}) \neq \varnothing$. Therefore by Theorem 4.5 (X,τ) is a δ-R_0 space.

Theorem 4.7. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) *is a* δ-R_0 *space;*

(2) *For any* $A \neq \varnothing$ *and* $G \in \delta O(X,\tau)$ *such that* $A \cap G \neq \varnothing$, *there exists* $F \in \delta C(X,\tau)$ *such that* $A \cap F \neq \varnothing$ *and* $F \subset G$*;*

(3) *Any* $G \in \delta O(X,\tau)$, $G = \cup\{F \in \delta C(X,\tau) \mid F \subset G\}$*;*

(4) *Any* $F \in \delta C(X,\tau), F = \cap\{G \in \delta O(X,\tau) \mid F \subset G\}$*;*

(5) *For any* $x \in X, Cl_\delta(\{x\}) \subset Ker_\delta(\{x\})$.

Proof. $(1) \to (2)$: Let A be a nonempty set of X and $G \in \delta O(X,\tau)$ such that $A \cap G \neq \varnothing$. There exists $x \in A \cap G$. Since $x \in G \in \delta O(X,\tau)$, $Cl_\delta(\{x\}) \subset G$. Set $F = Cl_\delta(\{x\})$, then $F \in \delta C(X,\tau)$, $F \subset G$ and $A \cap F \neq \varnothing$.

$(2) \to (3)$: Let $G \in \delta O(X,\tau)$, then $G \supset \cup\{F \in \delta C(X,\tau) \mid F \subset G\}$. Let x be any point of G. There exists $F \in \delta C(X,\tau)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \delta C(X,\tau) \mid F \subset G\}$ and hence $G = \cup\{F \in \delta C(X,\tau) \mid F \subset G\}$.

$(3) \to (4)$: This is obvious.

$(4) \to (5)$: Let x be any of x and $y \notin Ker_\delta(\{x\})$. There exists $V \in \delta O(X,\tau)$ such that $x \in V$ and $y \notin V$; hence $Cl_\delta(\{y\}) \cap V = \varnothing$. By (4) $(\cap\{G \in \delta O(X,\tau) \mid Cl_\delta(\{y\}) \subset G\}) \cap V = \varnothing$ and there exists $G \in \delta O(X,\tau)$ such that

$x \notin G$ and $Cl_\delta(\{y\}) \subset G$. Therefore, $Cl_\delta(\{x\}) \cap G = \emptyset$ and $y \notin Cl_\delta(\{x\})$. Consequently, we obtain $Cl_\delta(\{x\}) \subset Ker_\delta(\{x\})$.

$(5) \to (1)$: Let $G \in \delta O(X, \tau)$ and $x \in G$. Suppose $y \in Ker_\delta\{x\}$, then $x \in Cl_\delta\{y\}$ and $y \in G$. This implies that $Cl_\delta(\{x\}) \subset Ker_\delta(\{x\}) \subset G$. This shows that (X, τ) is a δ-R_0 space.

Corollary 4.8. *For a topological space* (X, τ), *the following properties are equivalent:*

(1) (X, τ) *is a* δ-R_0 *space;*

(2) $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$ *for all* $x \in X$.

Proof. $(1) \to (2)$: Suppose that (X, τ) is a δ-R_0 space. By Theorem 4.7, $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$ for each $x \in X$. Let $y \in Ker_\delta(\{x\})$, then $x \in Cl_\delta(\{y\})$ and by Theorem 4.5 $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore, $y \in Cl_\delta(\{x\})$ and hence $Ker_\delta(\{x\}) \subset Cl_\delta(\{x\})$. This shows that $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$.

$(2) \to (1)$: This is obvious by Theorem 4.7.

Theorem 4.9. *For a topological space* (X, τ), *the following properties are equivalent:*

(1) (X, τ) *is a* δ-R_0 *space;*

(2) $x \in Cl_\delta(\{y\})$ *if and only if* $y \in Cl_\delta(\{x\})$, *for any points x and y in X.*

Proof. $(1) \to (2)$: Assume that X is δ-R_0. Let $x \in Cl_\delta(\{y\})$ and D be any δ-open set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every δ-open set which contain *y* contains *x*. Hence $y \in Cl_\delta(\{x\})$.

$(2) \to (1)$: Let U be a δ-open set and $x \in U$. If $y \notin U$, then $x \notin Cl_\delta(\{y\})$ and hence $y \notin Cl_\delta(\{x\})$. This implies that $Cl_\delta(\{x\}) \subset U$. Hence (X, τ) is δ-R_0.

We observed that by Definition 9 and Theorem 4.9 the notions of δ-symmetric and δ-R_0 are equivalent.

Theorem 4.10. *For a topological space* (X, τ), *the following properties are equivalent:*

(1) (X, τ) *is a* δ-R_0 *space;*

(2) If F is δ*-closed, then* $F = Ker_\delta(F)$;

(3) If F is δ*-closed and* $x \in F$, *then* $Ker_\delta(\{x\}) \subset F$;

(4) If $x \in X$, *then* $Ker_\delta(\{x\}) \subset Cl_\delta(\{x\})$.

Proof. (1) $\to$ (2) : Let F be δ-closed and $x \notin F$. Thus $X - F$ is δ-open and contains x. Since (X,τ) *is* δ-R_0, $Cl_\delta(\{x\}) \subset X - F$. Thus $Cl_\delta(\{x\}) \cap F = \varnothing$ and by Lemma 3.1 $x \notin Ker_\delta(F)$. Therefore $Ker_\delta(F) = F$.

(2) $\to$ (3) : In general, $A \subset B$ implies $Ker_\delta(A) \subset Ker_\delta(B)$. Therefore, it follows from (2) that $Ker_\delta(\{x\}) \subset Ker_\delta(F) = F$.

(3) $\to$ (4) : Since $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is δ-closed, by (3) $Ker_\delta(\{x\}) \subset Cl_\delta(\{x\})$.

(4) $\to$ (1) : We show the implication by using Theorem 4.9. Let $x \in Cl_\delta(\{y\})$. Then by Lemma 4.1 $y \in Ker_\delta(\{x\})$. Since $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is δ-closed, by (4) we obtain $y \in Ker_\delta(\{x\} \subset Cl_\delta(\{x\})$. Therefore $x \in Cl_\delta(\{y\})$ implies $y \in Cl_\delta(\{x\})$. The converse is obvious and (X,τ) is δ-R_0.

Recall that a filterbase F is called *δ-convergent* to a point x in X, if for any δ-open set U of X containing x, there exists B in F such that B is a subset of U.

Lemma 4.11. *Let (X,τ) be a topological space and x and y any two points in X such that every net in X δ-converging to y δ-converges to x. Then $x \in Cl_\delta(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n\in N}$ is a net in $Cl_\delta(\{y\})$. Since $\{x_n\}_{n\in N}$ δ-converges to y, then $\{x_n\}_{n\in N}$ δ-converges to x and this implies that $x \in Cl_\delta(\{y\})$.

Theorem 4.12. *For a topological space (X,τ), the following statements are equivalent:*

(1) (X,τ) is a δ-R_0 space;

(2) If $x, y \in X$, then $y \in Cl_\delta(\{x\})$ if and only if every net in X δ-converging to y δ-converges to x.

Proof. (1) $\to$ (2) : Let $x, y \in X$ such that $y \in Cl_\delta(\{x\})$. Suppose that $\{x_\alpha\}_{\delta\in\Lambda}$ is a net in X such that $\{x_\alpha\}_{\delta\in\Lambda}$ δ-converges to y. Since $y \in Cl_\delta(\{x\})$, by Theorem 4.5 we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $x \in Cl_\delta(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-converges to x. Conversely, let $x, y \in X$ such that every net in X δ-converging to y δ-converges to x. Then $x \in Cl_\delta(\{y\})$ by Lemma 3.1. By Theorem 4.5, we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $y \in Cl_\delta(\{x\})$.

(2) $\to$ (1) : Assume that x and y are any two points of X such that $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) \neq \varnothing$. Let $z \in Cl_\delta(\{x\}) \cap Cl_\delta(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $Cl_\delta(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-converges to z. Since $z \in Cl_\delta(\{y\})$, then

$\{x_\alpha\}_{\alpha\in\Lambda}$ δ-converges to y. It follows that $y \in Cl_\delta(\{x\})$. By the same token we obtain $x \in Cl_\delta(\{y\})$. Therefore $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$ and by Theorem 4.5 (X, τ) is δ-R_0.

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