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An Example of a Probabilistic Metric Space Not Induced from a Random Normed Space

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Abstract. In this paper we present an example of a probabilistic metric space not induced from a random normed space.

1. Introduction and preliminaries

The notion of a probabilistic metric space was introduced by K. Menger and since then the theory of probabilistic metric spaces has developed in many directions [6]. In this paper we present an example of a probabilistic metric space not induced from a random normed space. The latter space was introduced by Serstnev [7]. We begin with some basic definitions.

A *distribution function* (briefly, a d.f.) is a function F from the extended real line $\overline{\mathsf{R}} = [-\infty, +\infty]$ into the unit interval $I = [0,1]$ that is nondecreasing and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$. We normalize all d.f.'s to be left-continuous on the unextended real line $\mathsf{R} = (-\infty, +\infty)$. The set of all d.f.'s will be denoted by Δ and the subset of those d.f.'s called distance d.f.'s. such that $F(0) = 0$, by Δ^+.

We introduce an ordering $'\leq'$ on Δ^+, by setting $F \leq G$ whenever $F(x) \leq G(x)$ for all F, G in Δ^+ and all x in the real line, R. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, viz. for all $F, G, H \in \Delta^+$, we have

$$\tau(\tau(F,G),H) = \tau(F,\tau(G,H)),$$
$$\tau(F,G) = \tau(G,F),$$
$$\tau(F,H) \le \tau(G,H) \text{ if } F \le G,$$
$$\tau(F,\varepsilon_0) = F.$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Typical continuous triangle functions are convolutions and the operations τ_T and τ_{T^*}, which are given by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),$$

and

$$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t)),$$

for all $F, G \in \Delta^+$ and all $x \in \mathbb{R}$ ([6] Secs.7.2 and 7.3). Here T is a continuous *t-norm*, i.e. a continuous binary operation on $[0,1]$ that is associative, commutative, non-decreasing and has 1 as identity; T^* is a continuous *t-conorm*, namely a continuous binary operation on $[0,1]$ that is related to the continuous *t*-norm T through

$$T^*(x,y) = 1 - T(1-x, 1-y).$$

Definition 1.1. [6] *A* probabilistic metric *(briefly PM) space is a triple* (S, f, τ), *where* S *is a nonempty set,* τ *is a triangle function, and* f *is a mapping from* $S \times S$ *into* Δ^+ *such that, if* F_{pq} *denote the value of* f *on the pair* (p,q), *the following hold for all* p, q, r *in* S :

(PM1) $F_{pq} = \varepsilon_0$ *if and only if* $p = q$.

(PM2) $F_{pq} = F_{qp}$.

(PM3) $F_{pr} \ge \tau(F_{pq}, F_{qr})$.

Definition 1.2. [7] *A* random normed space *is a triple* (V, v, τ), *where V is a vector space,* τ *is a continuous triangle function, and v is a mapping from V into* Δ^+ *such that the following hold for all* p, q, r *in* V :

(RN1) $\varepsilon_0 = v_p$ *if and only if* $p = \theta$, θ *being the null vector in V;*

(RN2) $v_{p+q} \geq \tau(v_p, v_q)$;

(RN3) $v_{\lambda p}(x) = v_p\left(\dfrac{x}{|\lambda|}\right)$ *for all x and* λ *in* $\mathbb{R}$.

We adopt the convention that $v_p(x/0) = \varepsilon_0(x)$.

It is easy to see that if $(V, \|\cdot\|)$ is a real normed space, if τ is a triangle function such that $\tau(\varepsilon_a, \varepsilon_b) \geq \varepsilon_{a+b}$ for all $a, b \geq 0$, and if $v : V \to \Delta^+$ is defined via $v_p = \varepsilon_{\|p\|}$ then (V, v, τ) is a random normed space.

On the other hand, if (V, v, τ) is a random normed space and if $f : V \times V \to \Delta^+$ is defined via $f(p, q) = F_{pq} = v_{p-q}$, for all p, q in V, then (V, f, τ) is a PM space. Indeed,

(PM1) $F_{pp} = v_{p-p} = v_\theta = \varepsilon_0$, $F_{pq} \neq \varepsilon_0$ where $p \neq q$ since $v_{p-q} \neq \varepsilon_0$;

(PM2) For any $x \in \mathbb{R}$,

$$F_{pq}(x) = v_{p-q}(x) = v_{-(q-p)}(x) = v_{q-p}\left(\frac{x}{|-1|}\right) = v_{q-p}(x) = F_{qp}(x);$$

(PM3) $\tau\left(F_{pq}, F_{qr}\right) = \tau\left(v_{p-q}, v_{q-r}\right) \leq v_{p-q+q-r} = v_{p-r} = F_{pr}$.

In this case, we say the PM space (V, f, τ) is *induced* by the random normed space (V, v, τ) and $f(p, q) = v_{p-q}$ is the induced metric.

Definition 1.3. *For any F in* Δ^+, *denote by* $F^\wedge$ *the left-continuous quasi-inverse of F, i.e., the function defined for all t in* $[0,1]$ *by*

$$F^\wedge(t) = \sup\left\{x \middle| F(x) < t\right\}.$$

Lemma 1.4. *Let* (V, v, t) *be a random normed space. For all x and* α *in* $\mathbb{R}$, *and* p *in* V *we have* $v^\wedge_{\alpha p}(x) = |\alpha| v^\wedge_p(x)$.

Proof. For all x and α in R, p in V and $v_{\alpha p}(x) = v_p(x/|\alpha|)$ then

$$
\begin{aligned}
v^{\wedge}_{\alpha p}(x) &= \sup\{t \,|\, v_{\alpha p}(t) < x\} \\
&= \sup\{t \,|\, v_p(t/|\alpha|) < x\}, \qquad y = t/|\alpha| \\
&= \sup\{|\alpha| y \mid v_p(y) < x\} \\
&= |\alpha| \sup\{y \mid v_p(y) < x\} \\
&= |\alpha| v^{\wedge}_p(x).
\end{aligned}
$$

So, $v^{\wedge}_{\alpha p}(x) = |\alpha| v^{\wedge}_p(x)$.

In order to know if a PM space is induced by a random normed space, we need the following theorem.

Theorem 1.5. *If* (V, F, τ) *is a PM space induced by a random normed space* (V, v, τ) *then* $F^{\wedge}_{(\alpha p)(\alpha q)}(x) = |\alpha| F^{\wedge}_{pq}(x)$, *for all* x *and* α *in* R, *and* p, q *in V.*

Proof. For all x and α in R, and p in V, we have, by Lemma 1.4, $v^{\wedge}_{\alpha p}(x) = |\alpha| v^{\wedge}_p(x)$. Then

$$
\begin{aligned}
F^{\wedge}_{(\alpha p)(\alpha q)}(x) &= \sup\{t \,|\, F_{(\alpha p)(\alpha q)}(t) < x\} = \sup\{t \,|\, v_{\alpha p - \alpha q}(t) < x\} \\
&= \sup\{t \,|\, v_{\alpha(p-q)}(t) < x\} = v^{\wedge}_{\alpha(p-q)}(x) = |\alpha| v^{\wedge}_{(p-q)}(x) \\
&= |\alpha| \sup\{t \,|\, v_{p-q}(t) < x\} = |\alpha| \sup\{t \,|\, F_{pq}(t) < x\} \\
&= |\alpha| F^{\wedge}_{pq}(x).
\end{aligned}
$$

2. The example

We have seen that a random normed space (V, v, τ) induces a PM space (V, f, τ) if $f(p, q) = F_{pq} = v_{p-q}$ for all p, q in V. The following example shows that the converse is not true, that is, not all PM spaces are induced by some random normed space. The idea is to modify the analogous procedure in classical functional analysis (see, e.g, [5], [8]).

Example 2.1. Let (V, f, τ) be a PM space that is induced by a random normed space. Define the d.f. F' by $F'_{pp} = \varepsilon_0$,

$$F'_{pq}(x) = \left\{ \begin{array}{ll} 0 & , \quad x \le 0 \\ \dfrac{F_{pq}(x)}{1 + F_{pq}(x)} & , \quad 0 < x < +\infty \\ 1 & , \quad x = +\infty \end{array} \right\},$$

for all p, q in V. Then (V, f', τ), where $f'(p,q) = F'_{pq}$, is a probabilistic metric space. We claim that this is not induced by a random normed space.

Firstly we show that (V, f', τ) is a PM space. We note that (PM1) and (PM2) are satisfied, because (V, f, τ) is a PM space. Obviously, $F_{pq}(x) < F_{pr}(x + y)$ and $F_{qr}(y) < F_{pr}(x + y)$ for all p, q, r in V, $0 < x \le +\infty$ and $0 < y \le +\infty$. We know the function $h(t) = \frac{t}{1+t}, t > -1$ is increasing so that

$$F'_{pq} = \frac{F_{pq}(x)}{1 + F_{pq}(x)} < \frac{F_{pr}(x + y)}{1 + F_{pr}(x + y)}, \; F'_{qr} = \frac{F_{qr}(y)}{1 + F_{qr}(y)} < \frac{F_{pr}(x + y)}{1 + F_{pr}(x + y)}.$$

Consequently, $\tau(F'_{pq}, F'_{qr}) \le \text{Min}(F'_{pq}, F'_{qr}) \le F'_{pr}$. If $F'_{pq} = 0$ or $F'_{qr} = 0$ or both of them are equal to zero, then $\tau(F'_{pq}, F'_{qr}) = 0 \le F'_{pr}$. Also if $F'_{pq} = 1$, $F'_{qr} = 1$ or both of them are equal to 1, then we have

$$\tau\left(F'_{pq}, F'_{qr}\right) \le \text{Min}\left(F'_{pq}, F'_{qr}\right) \le F'_{pr} = 1.$$

Hence (V, f', τ) is PM space.
For $0 < x \le +\infty$, by Definition 1.3, we note that

$$F'^{\wedge}_{pq}(x) = \sup\left\{ t \,\middle|\, F'_{pq}(t) < x \right\} = \sup\left\{ t \,\middle|\, \frac{F_{pq}(t)}{1 + F_{pq}(t)} < x \right\} = \frac{F^{\wedge}_{pq}(x)}{1 + F^{\wedge}_{pq}(x)}.$$

If (V, f', τ) is a PM space induced by some random normed space, then from Theorem 1.5 , we must have

$$\begin{aligned} F'^{\wedge}_{(\alpha p)(\alpha q)}(x) = |\alpha| F'^{\wedge}_{pq}(x) &\Rightarrow \frac{F^{\wedge}_{(\alpha p)(\alpha q)}(x)}{1 + F^{\wedge}_{(\alpha p)(\alpha q)}(x)} = |\alpha| \frac{F^{\wedge}_{pq}(x)}{1 + F^{\wedge}_{pq}(x)} \\ &\Rightarrow 1 + F^{\wedge}_{(\alpha p)(\alpha q)}(x) = 1 + F^{\wedge}_{pq}(x) \\ &\Rightarrow F^{\wedge}_{(\alpha p)(\alpha q)}(x) = F^{\wedge}_{pq}(x) \\ &\Rightarrow |\alpha| \, F^{\wedge}_{pq}(x) = F^{\wedge}_{pq}(x). \end{aligned}$$

That is only true when $\alpha = 1$, so that the PM space (V, f', τ) is not induced by a random normed space.

Most authors use the term "probabilistic normed space sense of Serstnev" instead of using the term "random normed space". Recently, C. Alsina, B. Schweizer and Sklar [1] (see also [3], [4]) gave a new definition of a probabilistic normed space. This definition, which is based on a characterization of normed space by means of a betweenness relation, includes the earlier definition of A.N. Serstnev as a special case. We will introduce the definition earlier mentioned.

Definition 2.1. *A* probabilistic normed *(briefly PN) space is a quadruple* (V, v, τ, τ^*), *where* V *is a vector space,* τ *and* τ^* *are continuous triangle functions with* $\tau \le \tau^*$, *and* v *is a mapping from* V *into* Δ^+ *such that, for all* p, q *in* V, *the following conditions hold:*

(PN1) $v_p = \varepsilon_0$ *if and only if* $p = \theta$, θ *is the null vector in* V*;*

(PN2) $v_{-p} = v_p$*;*

(PN3) $v_{p+q} \ge \tau(v_p, v_q)$ *;*

(PN4) $v_p \le \tau^*(v_{\alpha p}, v_{(1-\alpha)p})$, *for all* α *in* $[0,1]$.

If $\tau^* = \tau_M$ and equality holds in (PN4), then (V, v, τ, τ_M) is a Serstnev's PN space. In this case, as shown in [1]; the conditions

$$v_p = \tau_M(v_{\alpha p}, v_{(1-\alpha)p}), \quad \text{for all } p \text{ in } V \text{ and all } \alpha \text{ in } [0,1],$$

and (PN2), taken together, are equivalent to Serstnev's condition

$$v_{\lambda p}(x) = v_p\left(\frac{x}{|\lambda|}\right), \text{ for all } x \text{ and } \lambda \text{ in } \mathbb{R},$$

where, by convention, $v_p(x/0) = \varepsilon_0(x)$.

It is worth mentioning that these authors together with C. Sempi also came out with the notion of a probabilistic inner product space [2]. It is an open problem to find an example of a PM space which is not induced by a PN space.

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