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The Last Challenges and Open Questions of Professor M.R.M. Razali

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Abstract. Professor M.R.M. Razali visited Gunma University from February 10 – February 23, 2001 with his graceful wife. I was sad to learn that he passed away recently. I recall his good personality with profound love and sadness, and it reminded me of the open questions that we discussed during his visit to my university. His research interest was to examine the relationship between reproducing kernels and integral equations, from the viewpoint of numerical conformal mappings. As a tribute to his fond memory, I would like to delineate our open questions with the relevant backgrounds.

1. Introduction

In order to calculate the Riemann mapping function numerically, the theory of reproducing kernels (kernel functions) was developed long ago and a seminal book was published by S. Bergman [1]. Reproducing kernels are, in principle, computable, because reproducing kernels are represented by orthonormal complete systems in the reproducing kernel Hilbert spaces. The complete orthonormal systems can be constructed by the Gram-Schmidt orthogonalization procedure from complete and linearly independent functions in the spaces. S. Bergman and G. Szegö initiated their theories of reproducing kernels which are called, very popularly, the Bergman kernel and the Szegö kernel, respectively. Their kernels are very fundamental and have been developed extensively in complex analysis, quite apart from this initial idea, in both single and several complex analyses. Note here that the Riemann mapping function is simply represented by these typical reproducing kernels. Reproducing kernels are certainly computable in the above sense. However, more effective calculations were considered by combining the reproducing kernels and integral equations. That is, both reproducing kernels are the solutions of some Fredholm integral equations of the second kind. The second kind of integral equations can be solved numerically and the solutions obtained are computationally stable. Professor Razali was interested in this viewpoint and contributed to the interrelationship between the reproducing kernels and the associated integral equations, see for example, [2, 3, 4]. Of course, as his interest was in the Bergman kernel and the Szegö kernel, he considered reproducing kernels that are only typical in one complex variable for Bergman kernel and the Szegö kernel. Since he had two of my monographs [5, 6], he displayed keen interest in other reproducing kernels; that is, the Hardy H_2 kernel and the associated reproducing kernel. I think this might be a reason for his visit to our university. I introduced him these reproducing kernels and we had our new problems. I think these open problems, were the last challenges of Professor Razali. However, he wished to work on these problems in his country. Here, I would like to introduce these typical reproducing kernels in one variable complex analysis and our open questions, clearly, as my monument to his memory.

2. The Hardy H_2 kernel and its conjugate kernel

Let D denote a regular domain on the complex $z = x + iy$ plane whose boundary is a finite number of disjoint analytic Jordan curves. Let $G(z, t)$ denote the Green function on the domain D of the Laplace equation with a pole at t of D with the logarithmic singularity:

$$G(z, t) = \log \frac{1}{|z - t|} + \text{regular terms.}$$

Let $H_2(D)(H_2(D),$ respectively) denote the analytic (conjugate, respectively) Hardy space on D defined as the family $H_2(D)$ comprising analytic functions $f(z)$ on D with harmonic majorants $U(z)$ satisfying $|f(z)|^2 \leq U(z)$ on D and with finite norms

$$\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \frac{\partial G(z, t)}{\partial \nu_z} |dz| \right\}^{\frac{1}{2}} \tag{1}$$

$$\left(\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \left(\frac{\partial G(z, t)}{\partial \nu_z} \right)^{-1} |dz| \right\}^{\frac{1}{2}}, \text{ respectively} \right) \tag{2}$$

where $f(z)$ means Fatou's nontangential boundary value and $\partial / \partial \nu_z$ denotes the inner normal derivative with respect to D. Since $\partial G(z, t) / \partial \nu_z$ is a positive continuous function on ∂D, there exists, as in the Szegö kernel, the reproducing kernel $K_t(z, \overline{u})(\hat{K}_t(z, \overline{u}),$ respectively) such that

$$f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{K_t(z, \overline{u})} \frac{\partial G(z, t)}{\partial \nu_z} |dz| \tag{3}$$

$$\left(f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{\hat{K}_t(z,\overline{u})} \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz| , \quad \text{respectively} \right) \tag{4}$$

for all $u \in D$ and for all $f \in H_2(D)\,(\hat{H}_2(D)$, resp.). These reproducing kernels $K_t(z,\overline{u})$ and $\hat{K}_t(z,\overline{u})$ will be called the *Hardy* H_2 *kernel* and the *conjugate Hardy* H_2 *kernel* on (or, of) *D*, respectively.

For an arbitrary domain *S* on the complex plane and more generally, for any open Riemann surface *S*, we can define the Hardy space $H_2(S)$ and we can define the Hardy H_2 kernel by considering some regular exhaustion of *S* containing the point $t \in S$. Meanwhile, we can define the conjugate Hardy H_2 kernel on any compact bordered Riemann surface.

We see these Hardy H_2 kernel and conjugate kernel are typical and important in one variable complex analysis as the reproducing kernels of the third kind with the Bergman kernel and the Szegö kernel, from various viewpoints. See [5, 6] for the details. However, the structures of these kernels are much more involved than those of the Bergman and the Szegö kernels. So, apart from the viewpoint of numerical conformal mappings, we are interested in some effective calculations of these kernels. In particular, we wondered about these kernels as the solutions of some Fredholm integral equations of the second kind as in the Bergman and the Szegö kernels. These are our main open questions that we had discussed with Professor Razali in my University.

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