BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **26** (2003) 35–51

Numerical Conformal Mappings of Unbounded Multiply-Connected Domains Using the Charge Simulation Method

[1]KANAME AMANO, [1]DAI OKANO, [1]HIDENORI OGATA AND [2]MASAAKI SUGIHARA

[1]Department of Computer Science, Faculty of Engineering, Ehime University, Matsuyama 790-8577, Japan
[2]Department of Computational Science and Engineering, Graduate School of Engineering, Nagoya University, Nagoya 464-8603, Japan
[1]e-mail: amano@cs.ehime-u.ac.jp, okano@cs.ehime-u.ac.jp, ogata@cs.ehime-u.ac.jp and
[2]e-mail: sugihara@cse.nagoya-u.ac.jp

Abstract. We present a numerical method of conformal mappings of unbounded multiply-connected domains exterior to closed Jordan curves onto the three types of canonical domains of Nehari, i.e., the parallel slit domain, the circular slit domain and the radial slit domain. In the method, we express the mapping functions in terms of a pair of conjugate harmonic functions and approximate them, using the charge simulation method, by a linear combination of complex logarithmic functions. The method is simple without integration and suited for domains with curved boundaries. In particular, approximate mapping functions of an unbounded multiply-connected domain onto the three types of slit domains are obtained in a unified way by solving linear equations with a common coefficient matrix. A typical example shows the effectiveness of the method.

1. Introduction

The numerical conformal mapping has been an attractive subject in computational mathematics [6, 10, 14, 24]. We are here concerned with conformal mappings of an unbounded multiply-connected domain exterior to closed Jordan curves, which is shown in Figure 1, onto the three types of canonical domains of Nehari [17], i.e., (a) the parallel slit domain, (b) the circular slit domain and (c) the radial slit domain, which are shown in Figure 2. They are familiar in science and engineering, particularly in two-dimensional potential flow problems, though simple method of computation has not been available. We present a numerical method for computing these conformal mappings using the charge simulation method, which is simple without integration and suited for domains with curved boundaries. In particular, approximate mapping functions onto the three types of slit domains are obtained in a unified way by solving linear equations with a common coefficient matrix.

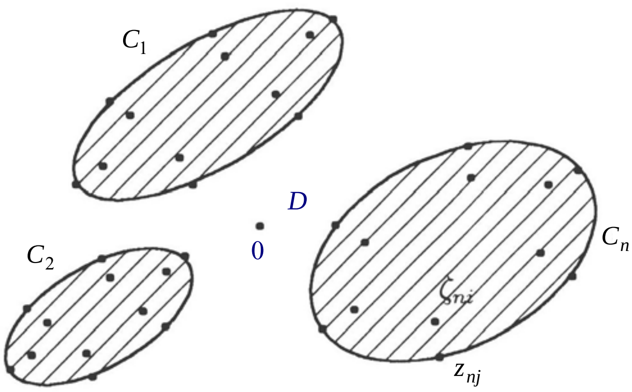
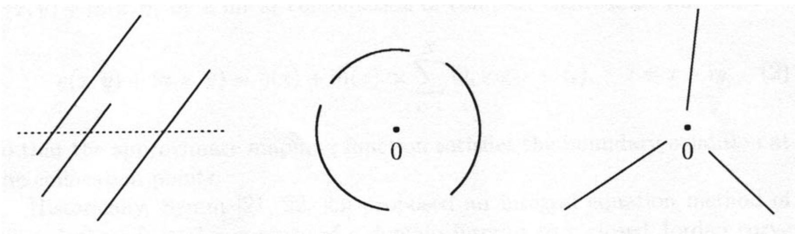


Figure 1. An unbounded multiply-connected problem domain in the *z*-plane, together with charge points and collocation points used in the charge simulation method.



(a) (b) (c)

Figure 2. The unbounded canonical domains of Nehari [17], i.e., (a) the parallel slit domain, (b) the circular slit domain and (c) the radical slit domain

The charge simulation method is originally a solver for the Laplace equation [13,15,16,18,20]. It approximates the solution by a linear combination of logarithmic potentials or fundamental solutions of the Laplace operator

$$g(x, y) = g(z) \approx \sum_{i=1}^{N} Q_i \log|z - \zeta_i| , \quad z = x + iy, \tag{1}$$

where Q_i are unknown real coefficients (called the *charges*) and ζ_i are given points (called the *charge points*) outside the problem domain. The unknown charges Q_i are determined from the *collocation condition,* i.e., the condition that the approximate solution (1) satisfies the boundary condition at a finite number of points (called the *collocation points*) placed on the boundary.

In our method, first we express the mapping function in terms of a pair of conjugate harmonic functions $g(x, y)$, $h(x, y)$. It is shown that the function $g(x, y)$ is subject to a boundary condition so that the boundary is mapped onto slits. Second we approximate the functions $g(x, y)$, $h(x, y)$ using the charge simulation method. Then we have an approximation of the function $g(x, y) + ih(x, y)$ by a linear combination of complex logarithmic functions

$$g(x, y) + ih(x, y) = g(z) + ih(z) \simeq \sum_{i=1}^{N} Q_i \log\left(z - \zeta_i\right), \quad z = x + iy \tag{2}$$

so that the approximate mapping function satisfies the boundary condition at the collocation points.

Historically, Symm [21, 22, 23] proposed an integral equation method of numerical conformal mappings of a domain interior to a closed Jordan curve onto the unit disk, a domain exterior to a closed Jordan curve onto the exterior of the unit disk, and a doubly-connected domain bounded by two closed Jordan curves onto a circular annulus. He expressed the mapping functions in terms of a single-layer logarithmic potential and reduced the problems to a singular Fredholm integral equation of the first kind. Gaier [7, 8] mathematically studied Symm's integral equation, and proved the existence and uniqueness of the solution. Symm approximated the source density by a step function in numerical computation, to which improvements have been made by using piecewise quadratic polynomials [9], cubic spline functions and singular functions [11, 12], trigonometric polynomials [19], etc. The charge simulation method [1, 2] is also regarded as a discretization of the source density using an auxiliary boundary [5]. This paper is a reformulation of our preceding works [3, 4].

2. Mapping theorems

Let D be an unbounded multiply-connected domain exterior to the closed Jordan curves $C_1, \cdots, C_n$ in the $z(= x + iy)$-plane shown in Figure 1. We suppose that the origin $z = 0$ is in D without loss of generality. We consider a conformal mapping of the domain D onto a parallel slit domain (Figure 2(a)), which is the entire $w(= u + iv)$-plane with parallel rectilinear slits[1]. The following theorem shows the existence of the conformal mapping [17].

[1]We suppose that the multiply-connected domain D and the slit domains appearing in this paper include the point at infinity.

Theorem 1. *For a given domain D and a given angle θ, there exists a unique analytic function $w = f_\theta(z)$ such that it (i) maps conformally the domain D onto a parallel slit domain whose slits form the angle θ with the real axis and (ii) satisfies $f_\theta(\infty) = \infty$ and has the Laurent expansion near $z = \infty$ of the form*

$$f_\theta(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{3}$$

The condition (ii) is called the *normalization condition*. One of the purposes of this paper is to give a numerical method for computing the conformal mapping $w = f_\theta(z)$ in Theorem 1.

In particular we are interested in the two cases: the conformal mapping $f_u(z) = f_{\pi/2}(z)$ onto a domain with slits parallel to the imaginary axis and the conformal mapping $f_v(z) = f_0(z)$ onto a domain with slits parallel to the real axis. The conformal mapping $f_\theta(z)$ in the general case is given in terms of them, i.e.,

$$f_\theta(z) = e^{i\theta}(\cos\theta f_v(z) - i \sin\theta f_u(z)). \tag{4}$$

The conformal mapping $w = f_u(z)$ maps the boundary curves $C_1, \cdots, C_n$ onto the slits parallel to the imaginary axis, i.e.,

$$\mathrm{Re}\, f_u(z) = u_m , \quad z \in C_m , \; m = 1, \cdots, n, \tag{5}$$

and the conformal mapping $w = f_v(z)$ maps the boundary curves $C_1, \cdots, C_n$ onto the slits parallel to the real axis, i.e.,

$$\mathrm{Im}\, f_v(z) = v_m , \quad z \in C_m , \; m = 1, \cdots, n, \tag{6}$$

where u_m and v_m are constants indicating the positions of the slits.

In addition, we consider a conformal mapping of the domain D onto a circular slit domain (Figure 2(b)), which is the entire w-plane with circular slits whose common center is the origin. We also consider a conformal mapping of D onto a radial slit domain (Figure 2(c)), which is the entire w-plane with radial slits pointing at the origin. The following theorems show the existence of the conformal mappings [17].

Theorem 2. *For a given domain D, there exists a unique analytic function $w = f_c(z)$ such that it (i) maps conformally the domain D onto a circular slit domain and (ii) satisfies $f_c(0) = 0$, $f_c(\infty) = \infty$ and has the Laurent expansion near $z = \infty$ of the form*

$$f_c(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots . \tag{7}$$

Theorem 3. *For a given domain D, there exists a unique analytic function* $w = f_r(z)$ *such that it (i) maps conformally the domain D onto a radial slit domain and (ii) satisfies* $f_r(0) = 0$, $f_r(\infty) = \infty$ *and has the Laurent expansion near* $z = \infty$ *of the form*

$$f_r(z) = z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots. \tag{8}$$

The condition (ii) in Theorems 2 and 3 is also called the normalization condition. The conformal mapping $w = f_c(z)$ maps the boundary curves $C_1, \cdots, C_n$ onto the circular slits, i.e.,

$$|f_c(z)| = r_m, \quad z \in C_m, \quad m = 1, \cdots, n, \tag{9}$$

and the conformal mapping $w = f_r(z)$ maps the boundary curves $C_1, \cdots, C_n$ onto the radial slits, i.e.,

$$\arg f_r(z) = \theta_m, \quad z \in C_m, \; m = 1, \cdots, n, \tag{10}$$

where r_m and θ_m are constants indicating the radii and the arguments of the slits, respectively.

The mapping functions $w = f_u(z)$, $f_v(z)$, $f_c(z)$, $f_r(z)$ are treated in a unified way as described in the followings. We express the mapping functions as

$$f_u(z) = z + g_u(z) + ih_u(z), \tag{11}$$

$$f_v(z) = z + i(g_v(z) + ih_v(z)), \tag{12}$$

$$f_c(z) = z\exp(g_c(z) + ih_c(z)), \tag{13}$$

$$f_r(z) = z\exp(i(g_r(z) + ih_r(z))), \tag{14}$$

where $(g_u(z), h_u(z))$, $(g_v(z), h_v(z))$, $(g_c(z), h_c(z))$, $(g_r(z), h_r(z))$ are pairs of conjugate harmonic functions in *D*. Then, from the boundary conditions (5), (6), (9), (10), we have

$$g_u(z) = u_m - x, \qquad z \in C_m, \; m = 1, \cdots, n, \tag{15}$$

$$g_v(z) = v_m - y, \qquad z \in C_m, \; m = 1, \cdots, n, \tag{16}$$

$$g_c(z) = \log r_m - \log|z|, \quad z \in C_m, \; m = 1, \cdots, n, \tag{17}$$

$$g_r(z) = \theta_m - \arg z, \qquad z \in C_m, \; m = 1, \cdots, n, \tag{18}$$

Omitting the suffices of the functions $g_u(z)$, $g_v(z)$, $g_c(z)$, $g_r(z)$, $h_u(z)$, $h_v(z)$, $h_c(z)$, $h_r(z)$, these conditions are rewritten into the unified form

$$g(z) = s_m - t(z), \quad z \in C_m\,, \; m = 1, \cdots, n\,, \tag{19}$$

where

$$s_m = \begin{cases} u_m \\ v_m \\ \log r_m \\ \theta_m \end{cases} \quad \text{and} \quad t(z) = \begin{cases} x = \operatorname{Re} z & \text{if } g(z) = g_u(z) \\ y = \operatorname{Im} z & \text{if } g(z) = g_v(z) \\ \log|z| & \text{if } g(z) = g_c(z) \\ \arg z & \text{if } g(z) = g_r(z) \end{cases}. \tag{20}$$

Besides, from the normalization conditions (3), (7), (8), we have

$$g(\infty) + ih(\infty) = 0\,. \tag{21}$$

The conditions $f_c(0)$, $f_r(0) = 0$ are obviously satisfied from the expressions (13), (14). Conversely, if the pair of functions $(g(z), h(z)) = (g_u(z), h_u(z))$, $(g_v(z), h_v(z))$, $(g_c(z), h_c(z))$, $(g_r(z), h_r(z))$ satisfy the conditions (19), (21), the functions $f_u(z)$, $f_v(z)$, $f_c(z)$, $f_r(z)$ respectively given by (11), (12), (13), (14) are the mapping functions in question. Therefore, from the unique existence of the mapping functions, our problems are reduced to the problem of finding a pair of conjugate harmonic functions $(g(z), h(z))$ subject to the conditions (19), (21) together with the constants s_m, $m = 1, \cdots, n$ indicating the positions of the slits.

3. Numerical method

We intend to obtain approximate mapping functions by applying the charge simulation method to the pair of conjugate harmonic functions $(g(z), h(z))$. The charge simulation method here approximates the function $g(z)$ by

$$g(z) \approx G(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log|z - \zeta_{\ell i}| + (\text{constant}), \tag{22}$$

where $Q_{\ell i}$ are unknown real coefficients (called the *charges*) and $\zeta_{\ell i}$ are given points (called the *charge points*) placed inside the boundary curves C_ℓ. Then we can approximate the function $h(z)$, which is the conjugate harmonic function of $g(z)$, by

$$h(z) \approx H(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \arg(z - \zeta_{\ell i}) + (\text{constant}). \tag{23}$$

Consequently we have the approximation

$$g(z) + ih(z) \simeq G(z) + iH(z) = Q_0 + \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log\left(z - \zeta_{\ell i}\right), \tag{24}$$

where Q_0 is a complex constant.

The unknown constants Q_0, $Q_{\ell i}$, $i = 1, \cdots, N_\ell$, $\ell = 1, \cdots, n$ are determined from the following three conditions.

(i) *Single-valuedness condition:* It is naturally required that the approximate mapping function is single-valued in the problem domain *D*. This condition is equivalent to

$$\sum_{i=1}^{N_\ell} Q_{\ell i} = 0, \quad \ell = 1, \cdots, n. \tag{25}$$

In fact, the single-valuedness condition is equivalent to the condition that, for any closed contour $\tilde{C}_\ell$ surrounding each boundary curve C_ℓ, the relation

$$\int_{\tilde{C}_\ell} dH(z) = 0 \tag{26}$$

holds good. Since we have

$$\int_{\tilde{C}_\ell} dH(z) = \int_{\tilde{C}_\ell} d \sum_{m=1}^{n} \sum_{i=1}^{N_m} Q_{mi} \arg(z - \zeta_{mi}) = 2\pi \sum_{i=1}^{N_\ell} Q_{\ell i}$$

from (23), the single-valuedness condition is equivalent to (25).

(ii) *Normalization condition:* It is also naturally required that the approximate function $G(z) + iH(z)$ satisfies the normalization condition (21), i.e.,

$$G(\infty) + iH(\infty) = 0. \tag{27}$$

Remarking that

$$G(z) + iH(z) \simeq Q_0 + \left(\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \right) \log z \quad (z \to \infty),$$

we have

$$Q_0 = 0 \tag{28}$$

if the single-valuedness condition (25) is satisfied. Therefore the approximation formula (24) is rewritten into

$$g(z) + ih(z) \approx G(z) + iH(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log(z - \zeta_{\ell i}), \tag{29}$$

and the charges $Q_{\ell i}$, $i = 1, \cdots, N_\ell$, $\ell = 1, \cdots, n$, together with the constants s_m, $m = 1, \cdots, n$, remain unknown.

(iii) *Collocation condition:* We suppose that the approximate function $G(z)$ satisfies the boundary condition (19) at a finite number of points, i.e.,

$$G(z_{mj}) = S_m - t_{mj}, \quad z_{mj} \in C_m,\ j = 1, \cdots, N_m,\ m = 1, \cdots, n, \tag{30}$$

where z_{mj} are given boundary points (called the *collation points*), S_m are approximate values of s_m and $t_{mj} = t(z_{mj})$. The equalities (30) are rewritten into

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log\left| z_{mj} - \zeta_{\ell i} \right| - S_m = -t_{mj},$$

$$z_{mj} \in C_m,\ j = 1, \cdots, N_m,\ m = 1, \cdots, n. \tag{31}$$

The equalities (25), (31) constitute $N_1 + \cdots + N_n + n$ simultaneous linear equations for $N_1 + \cdots + N_n + n$ unknowns $Q_{\ell i}$, $i = 1, \cdots, n_\ell$, $\ell = 1, \cdots, n$ and S_m, $m = 1, \cdots, n$. We determine the unknown constants $Q_{\ell i}$ and S_m by solving these simultaneous linear equations and obtain the approximate function $G(z) + iH(z)$ by (29), which gives an approximate mapping function by substituting $g(z) + ih(z) \approx G(z) + iH(z)$ into (11), (12), (13), (14).

In practical computations, however, we cannot use the approximation formula (29) as it is because of the following reason. We usually employ the principal value of the logarithmic function $\operatorname{Log} z$, the branch of $\log z$ such that $-\pi < \arg z \le \pi$, for computing $\log z$. Then each term of $\log(z - \zeta_{\ell i})$ on the right-hand side of (29) has a discontinuity of $2\pi i$ on the half-infinite straight line parallel to the real axis $(-\infty + i \operatorname{Im} \zeta_{\ell i}]$, which causes discontinuities of the function $G(z) + iH(z)$ in the domain D. Therefore we have to rewrite the expression (29) into forms which are mathematically equivalent to (29) and are continuous in the domain D even if the principal value $\operatorname{Log} z$ is employed for computing $\log z$. We will call approximate mapping functions using rewritten expressions of $G(z) + iH(z)$ satisfying the above properties *continuous schemes*.

3.1. Starlike case

If each boundary curve C_ℓ , $\ell = 1, \cdots, n$ is starlike with respect to its inside point $\zeta_{\ell 0}$, remarking (25), we subtract $0 = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log(z - \zeta_{\ell 0})$ from the both sides of (29) and obtain the expression

$$G(z) + iH(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log\left(\frac{z - \zeta_{\ell i}}{z - \zeta_{\ell 0}}\right). \qquad (32)$$

The term of $\log((z - \zeta_{\ell i})/(z - \zeta_{\ell 0}))$ on the right-hand side of (32) has a discontinuity on a finite straight line connecting the two points $\zeta_{\ell 0}$ and $\zeta_{\ell i}$, which is included inside the boundary curve C_ℓ . Therefore the expression (32) is continuous using the principal value of logarithmic function. Consequently we have the following algorithm.

Algorithm (Continuous scheme) 1. *Suppose that each boundary curve C_ℓ , $\ell = 1, \cdots, n$ of the domain D is starlike with respect to its inside point $\zeta_{\ell 0}$. We intend to find approximate mapping functions in the forms*

$$f_u(z) \circledast F_u(z) = z + G_u(z) + iH_u(z), \qquad (33)$$

$$f_v(z) \circledast F_v(z) = z + i(G_v(z) + iH_v(z)), \qquad (34)$$

$$f_c(z) \circledast F_c(z) = z \exp(G_c(z) + iH_c(z)), \qquad (35)$$

$$f_r(z) \circledast F_r(z) = z \exp(i(G_r(z) + iH_r(z))), \qquad (36)$$

where $(G_u(z), H_u(z))$, $(G_v(z), H_v(z))$, $(G_c(z), H_c(z))$, $(G_r(z), H_r(z))$ *are pairs of conjugate harmonic functions in D. Then, omitting the suffices of* $(G_u(z), H_u(z))$, *etc., the pair of conjugate harmonic functions are given by*

$$G(z) + iH(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log\left(\frac{z - \zeta_{\ell i}}{z - \zeta_{\ell 0}}\right), \qquad (37)$$

where $\zeta_{\ell i}$, $i = 1, \cdots, N_\ell$, $\ell = 1, \cdots, n$ *are given charge points inside* C_ℓ *and* $Q_{\ell i}$, $i = 1, \cdots, N_\ell$, $\ell = 1, \cdots, n$ *are unknown real coefficients. The unknown coefficients, the charges,* $Q_{\ell i}$ *are determined, together with the unknown constants* S_m $(\circledast s_m)$, $m = 1, \cdots, n$ *indicating the positions of the slits, by the simultaneous linear equations*

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell i} \log \left| \frac{z_{mj} - \zeta_{\ell i}}{z_{mj} - \zeta_{\ell 0}} \right| - S_m = -t_{mj} \,,$$

$$z_{mj} \in C_m \,, \quad j = 1, \cdots, N_m \,, \quad m = 1, \cdots, n \,, \tag{38}$$

$$\sum_{i=1}^{N_\ell} Q_{\ell i} = 0 \,, \quad \ell = 1, \cdots, n \,, \tag{39}$$

where z_{mj} *are given collocation points on* C_m *and*

$$t_{mj} = t(z_{mj}) = \begin{cases} x_{mj} = \operatorname{Re} z_{mj} & \text{if } (G(z), H(z)) = (G_u(z), H_u(z)) \\ y_{mj} = \operatorname{Im} z_{mj} & \text{if } (G(z), H(z)) = (G_v(z), H_v(z)) \\ \log |z_{mj}| & \text{if } (G(z), H(z)) = (G_c(z), H_c(z)) \\ \arg z_{mj} & \text{if } (G(z), H(z)) = (G_r(z), H_r(z)) \end{cases} . \tag{40}$$

Besides, using the constants S_m *obtained by solving the linear equations,*

(1) *in the case of the mapping* $w = F_u(z)$, *each slit parallel to the imaginary axis lies approximately on* $\operatorname{Re} w = U_m$, $U_m = S_m$, $m = 1, \cdots, n$,

(2) *in the case of the mapping* $w = F_v(z)$, *each slit parallel to the real axis lies approximately on* $\operatorname{Im} w = V_m$, $V_m = S_m$, $m = 1, \cdots, n$,

(3) *in the case of the mapping* $w = F_c(z)$, *each circular slit lies approximately on* $|w| = R_m$, $\log R_m = S_m$, $m = 1, \cdots, n$,

(4) *in the case of the mapping* $w = F_r(z)$, *each radial slit lies approximately on* $\arg w = \theta_m$, $\theta_m = S_m$, $m = 1, \cdots, n$.

3.2. General case

In general cases where the boundary curves $C_1, \cdots, C_n$ are not necessarily starlike, remarking (25), we change the unknowns from the charges $Q_{\ell i}$ to their sums Q_ℓ^i such that

$$Q_\ell^i = \sum_{k=1}^{i} Q_{\ell k} \,, \quad i = 1, \cdots, N_\ell - 1, \ \ell = 1, \cdots, n \,, \tag{41}$$

$$Q_\ell^{N_\ell} = 0 \,, \quad \ell = 1, \cdots, n \,, \tag{42}$$

and rewrite (29) into the following expression

$$\begin{aligned} G(z) + iH(z) &= \sum_{\ell=1}^{n} \left\{ Q_\ell^1 \log(z - \zeta_{\ell 1}) + \sum_{i=2}^{N_\ell} (Q_\ell^i - Q_\ell^{i-1}) \log(z - \zeta_{\ell i}) \right\} \\ &= \sum_{\ell=1}^{n} \left\{ \sum_{i=1}^{N_\ell - 1} Q_\ell^i \left(\log(z - \zeta_{\ell i}) - \log(z - \zeta_{\ell i+1})\right) + Q_\ell^{N_\ell} \log(z - \zeta_{\ell N_\ell}) \right\} \\ &= \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i \log\left(\frac{z - \zeta_{\ell i}}{z - \zeta_{\ell i+1}} \right). \end{aligned} \tag{43}$$

The term of $\log((z - \zeta_{\ell i})/(z - \zeta_{\ell i+1}))$ on the right-hand side of (43) has a discontinuity on a finite straight line connecting the two charge points $\zeta_{\ell i}$ and $\zeta_{\ell i+1}$, which is included inside the boundary curve C_ℓ if the charge points are so placed that the straight line does not intersect C_ℓ. Then the expression (43) is continuous using the principal value of the logarithmic function. Consequently we have the following algorithm.

Algorithm (Continuous scheme) 2. *We also intend to find approximate mapping functions in the same forms as* (33), (34), (35), (36) *in Algorithm* 1. *In this case, the pair of conjugate harmonic functions are given by*

$$G(z) + iH(z) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i \log\left(\frac{z - \zeta_{\ell i}}{z - \zeta_{\ell i+1}} \right), \tag{44}$$

where $\zeta_{\ell i}$, $i = 1, \cdots, N_\ell$, $\ell = 1, \cdots, n$ *are given charge points inside* C_ℓ *and* Q_ℓ^i, $i = 1, \cdots, N_\ell - 1$, $\ell = 1, \cdots, n$ *are unknown real coefficients. The unknown coefficients, sums of the charges,* Q_ℓ^i *are determined, together with the unknown constants* $S_m (\circledast s_m)$, $m = 1, \cdots, n$ *indicating the positions of the slits, by the* $N_1 + \cdots + N_n$ *simultaneous linear equations*

$$\begin{aligned} &\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i \log\left| \frac{z - \zeta_{\ell i}}{z - \zeta_{\ell i+1}} \right| - S_m = -t_{mj}, \\ &z_{mj} \in C_m, \ j = 1, \cdots, N_m, \ m = 1, \cdots, n, \end{aligned} \tag{45}$$

where z_{mj} *are given collocation points on* C_m *and* t_{mj}, S_m *are the same as those in Algorithm* 1.

Algorithms 1 and 2 give the same results if they are applied to the problem domain whose boundary curves are starlike as stated previously.

From the maximum modulus theorem for analytic functions and the normalization conditions for the pair of conjugate harmonic functions (21), (27), though the problem domain D is not bounded, the absolute error of the approximate mapping functions onto the parallel slit domains, i.e.,

$$E_u(z) = \left| F_u(z) - f_u(z) \right| = \left| G_u(z) - g_u(z) + i(H_u(z) - h_u(z)) \right|, \tag{46}$$

$$E_v(z) = \left| F_v(z) - f_v(z) \right| = \left| G_v(z) - g_v(z) + i(H_v(z) - h_v(z)) \right|, \tag{47}$$

and the relative error of the approximate mapping functions onto the circular and the radial slit domains, i.e.,

$$E_c(z) = \left| \frac{F_c(z) - f_c(z)}{f_c(z)} \right| = \left| \exp(G_c(z) - g_c(z) + i(H_c(z) - h_c(z))) - 1 \right|, \tag{48}$$

$$E_r(z) = \left| \frac{F_r(z) - f_r(z)}{f_r(z)} \right| = \left| \exp\left\{ i(G_r(z) - g_r(z) + i(H_r(z) - h_r(z))) \right\} - 1 \right|, \tag{49}$$

take their maximum values somewhere on the boundary curves.

Once the approximate mapping functions $F_u(z)$, $F_v(z)$ are obtained, from (4), the approximate mapping function in the general case $F_\theta(z)$ is given by

$$F_\theta(z) = e^{i\theta}(\cos\theta F_v(v) - i\sin\theta\, F_u(z)) \tag{50}$$

for any θ without solving another set of linear equations; and the error is estimated by

$$\begin{aligned} E_p(z) &= \left| F_\theta(z) - f_\theta(z) \right| \\ &\le \left| F_u(z) - f_u(z) \right| + \left| F_v(z) - f_v(z) \right| = E_u(z) + E_v(z). \end{aligned} \tag{51}$$

4. An example

The problem domain is the exterior to three circles with different radii, i.e.,

$$D : \left| z - \zeta_{\ell 0} \right| > \rho_\ell, \quad \zeta_{\ell 0} = 2\exp\frac{2(\ell - 1)\pi i}{3}, \ \ell = 1, 2, 3,$$
$$\rho_1 = 1,\ \rho_2 = 0.5,\ \rho_3 = 1.5.$$

Table 1. Numerical results of the conformal mappings $w = F_u(z), F_v(z), F_c(z), F_r(z)$ ($cond$ = 4.7E4, $N = 64$, $q = 0.8$).

	$E_{u\ell}$	U_ℓ	$E_{v\ell}$	V_ℓ
C_1	7.1E-8	1.32203175	5.0E-8	−0.326576607
C_2	2.2E-8	−0.787052952	3.0E-8	0.99641470
C_3	1.2E-7	−0.697257057	8.1E-8	−1.478338666

	$E_{c\ell}$	R_ℓ	$E_{r\ell}$	θ_ℓ
C_1	2.1E-7	2.69585239	6.7E-8	−0.23582973
C_2	3.9E-8	2.91217882	2.1E-8	2.246730504
C_3	8.6E-5	2.265373689	2.5E-5	−2.005025898

Collocation points and charge points are placed by

$$z_{\ell j} = \zeta_{\ell 0} + \rho_\ell \exp\frac{2(j-1)\pi i}{N}, \quad \zeta_{\ell j} = \zeta_{\ell 0} + q\rho_\ell \exp\frac{2(j-1)\pi i}{N}, \quad j = 1, \cdots, N, \quad \ell = 1, 2, 3,$$

where $0 < q < 1$ is a parameter for charge placement. Since exact solutions are unknown, deviations of the approximate mapping functions from the slits are computed as an indication of errors, i.e.,

$$E_{u\ell} = \max_{1 \le j \le N} \left| \operatorname{Re} F_u(z_{\ell j+1/2}) - U_\ell \right|,$$

$$E_{v\ell} = \max_{1 \le j \le N} \left| \operatorname{Im} F_v(z_{\ell j+1/2}) - V_\ell \right|,$$

$$E_{c\ell} = \max_{1 \le j \le N} \left| |F_c(z_{\ell j+1/2})| - R_\ell \right|,$$

$$E_{r\ell} = \max_{1 \le j \le N} \left| \arg F_r(z_{\ell j+1/2}) - \theta_\ell \right|, \quad z_{\ell j+1/2} \in C_\ell, \quad \ell = 1, 2, 3,$$

where $z_{\ell j+1/2}$ is the middle point on C_ℓ between the two successive collocation points $z_{\ell j}$ and $z_{\ell j+1}$, and $z_{\ell N+1} = z_{\ell 1}$.

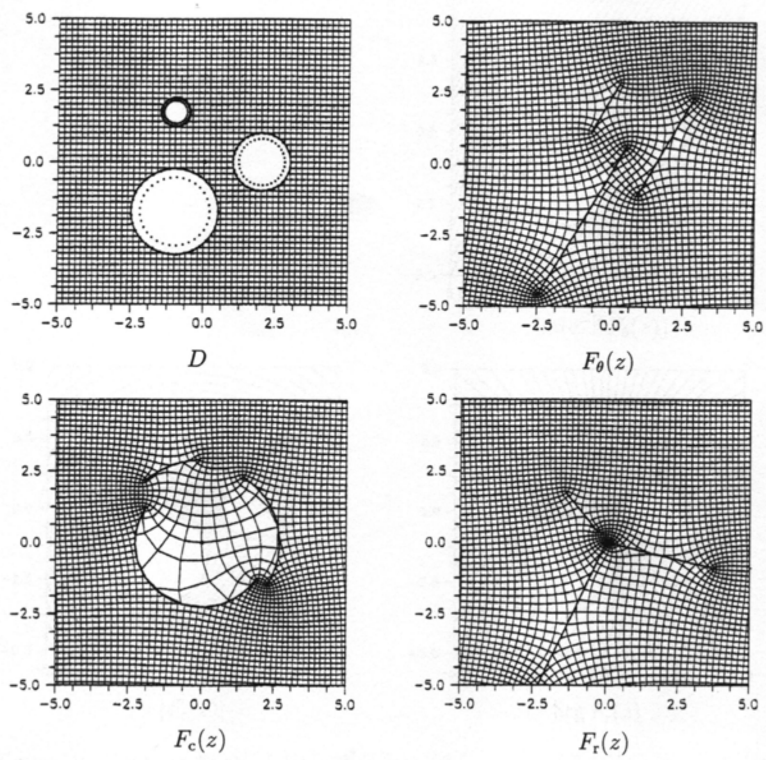


Figure 3. Numerical conformal mappings of the domain D exterior to three circles in the z-plane onto a parallel slit domain by $w = F_\theta(z)$ $(\theta = \pi/3)$, onto a circular slit domain by $w = F_c(z)$ and onto a radial slit domain by $w = F_r(z)$.

Figure 3 and Table 1 show the numerical results, where *cond* is the L_1 condition number of the coefficient matrix of the linear equations to be solved. The values of $U_\ell, V_\ell, R_\ell, \theta_\ell$ are shown until a nonzero digit appears in $U_\ell^{(N)} - U_\ell^{(2N)}$, etc., where (N) and $(2N)$ mean the numbers of simulation charges used. High accuracy is obtained, which is naturally expected from the fact that the charge simulation method gives a highly accurate result for domains with circular boundary curves[2].

Figure 4 shows contour lines of $\mathrm{Im}(e^{-i\theta}F_\theta(z))$ $(\theta = \pi/3)$, $|F_c(z)|$ and $\arg F_r(z)$. They illustrate the streamlines of a uniform flow, a vortex flow and a point-source (or sink) flow around obstacles. A vortex or a point-source is located at the origin, and the disks are cross sections of cylindrical objects.

[2]In general, it is known that, if boundary curves and boundary values are analytic, error in the charge simulation method decays exponentially as a function of the number of simulation charges [13].

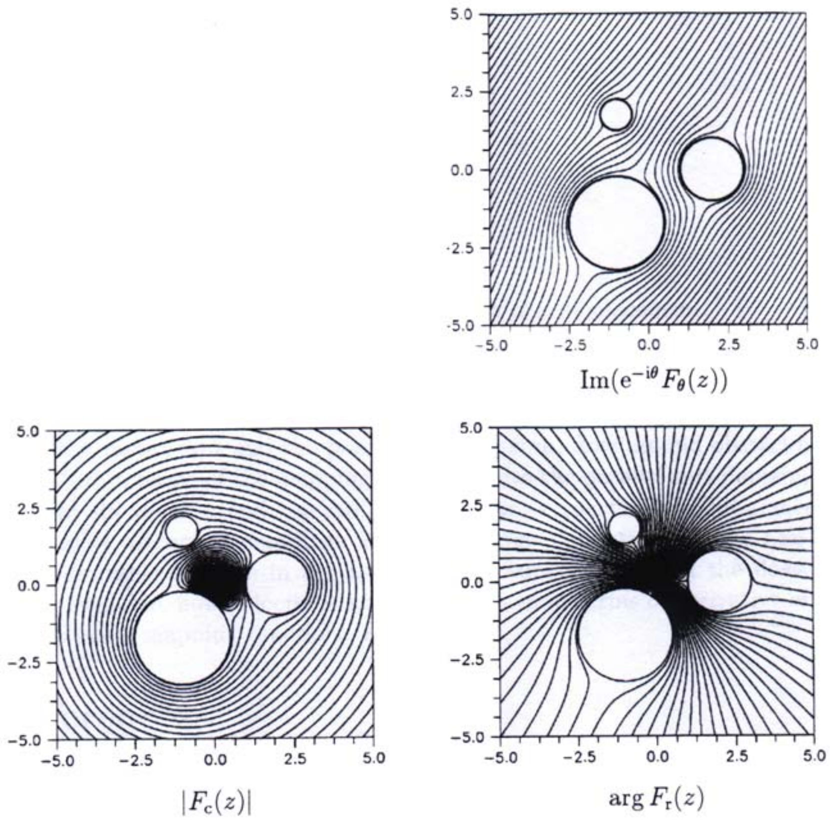


Figure 4. Contour lines of $\mathrm{Im}(e^{-i\theta}F_{\theta}(z))$ $(\theta = \pi/3)$, $\left|F_c(z)\right|$ and $\arg F_r(z)$, which illustrate the streamlines of a uniform flow, a vortex flow and a point-source (or sink) flow around obstacles, respectively.

Charge points as well as collocation points play an important role in the charge simulation method. The optimal arrangement is still an open problem. However, a practical method is available for charge placement in the case of problem domains with non-circular boundary curves, which is given by

$$\zeta_{\ell j} = z_{\ell j} + iq\,(z_{\ell j+1} - z_{\ell j-1}), \quad j = 1, \cdots, N_{\ell}, \quad \ell = 1, \cdots, n$$

as shown in Figure 5, where $z_{\ell 0} = z_{\ell N_{\ell}}$, $z_{\ell N_{\ell}+1} = z_{\ell 1}$, and $q > 0$ is a parameter. It is easy to find a nearly optimal value of q since the error first decreases exponentially as a function of q, and then turns to increase. See [1, 2, 3, 4] for numerical examples of conformal mappings of various shapes of problem domains by the charge simulation method. High accuracy is obtained if the problem domain has no reentrant corners.

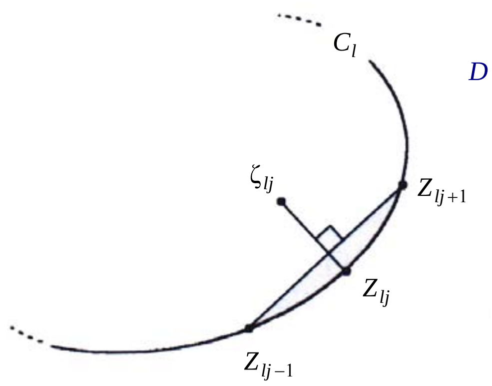


Figure 5. A practical method for charge placement.

5. Concluding remarks

We have presented a numerical method of conformal mappings of unbounded multiply-connected domains exterior to closed Jordan curves onto the three types of slit domains, i.e., the parallel slit domain, the circular slit domain and the radial slit domain. The method is simple without integration, and suited for domains with curved boundaries. In particular, the approximate mapping functions of a problem domain onto the three types of slit domains are obtained in a unified way by solving linear equations with a common coefficient matrix. The two schemes of approximate mapping functions are continuous and analytic as they are using the principal value of logarithmic function in practical computation.

These conformal mappings are familiar as a mathematical tool in two-dimensional potential flow analysis. Complex flow potentials of the uniform flow, the vortex flow and the point-source (or sink) flow around obstacles are described in terms of the mapping functions onto the parallel slits domain, the circular slit domain and the radial slit domain, respectively. The numerical method of conformal mappings using the charge simulation method also has some desirable features in applications, e.g., super positions of the flows are easily available, flow velocity is analytically given in terms of derivative of the approximate mapping functions, etc.

References

1. K. Amano, A bidirectional method for numerical conformal mapping based on the charge simulation method, *J. Inform. Process.* **14** (1991), 473–482.
2. K. Amano, A charge simulation method for the numerical conformal mapping of interior, exterior and doubly-connected domains, *J. Comput. Appl. Math.* **53** (1994), 353–370.
3. K. Amano and M. Sugihara, Numerical conformal mapping onto parallel slit domains with application to potential flow analysis in: G. Yagawa and C. Miki, Eds., *Proc 46th NCTAM, Theoretical and Applied Mechanics* **46** (Hokusen-sha, Tokyo, 1997), 295–305.

4. K. Amano, A charge simulation method for numerical conformal mapping onto circular and radial slit domains, *SIAM J. Sci. Comput.* **19** (1998), 1169–1187.
5. G. Fairweather and R.L. Johnston, The method of fundamental solutions for problems in potential theory, in: C.T.H. Baker and G.F. Miller, Eds., *Treatment of Integral Equations by Numerical Methods* (Academic Press, London, 1982), 349–359.
6. D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, Berlin, 1964.
7. D. Gaier, Integralgleichungen erster Art und konforme Abbildung, *Math. Z.* **147** (1976), 113–129.
8. D. Gaier, Das logarithmische Potential und die konforme Abbildung mehrfach zusammenhängender Gebiete, in: P.L. Butzer and F. Fehér, Eds., *E.B. Christoffel, The Influence of his Work on Mathematics and the Physical Sciences* (Birkhäuser, Basel, 1981), 290–303.
9. J.K. Hayes, D.K. Kahaner and R.G. Kellner, An improved method for numerical conformal mapping, *Math. Comput.* **26** (1972), 327–334.
10. P. Henrici, *Applied and Computational Complex Analysis,* Vol. 3, Wiley, New York, 1986.
11. D.M. Hough and N. Papamichael, The use of splines and singular functions in an integral equation method for conformal mapping, *Numer. Math.* **37** (1981), 133–147.
12. D.M. Hough and N. Papamichael, An integral equation method for the numerical conformal mapping of interior, exterior and doubly-connected domains, *Numer. Math.* **41** (1983), 287–307.
13. M. Katsurada and H. Okamoto, The collocation points of the fundamental solution method for the potential problem, *Computers Math. Applic.* **31** (1996), 123–137.
14. P.K. Kythe, *Computational Conformal Mapping,* Birkhäuser, Boston, 1998.
15. S. Murashima, *Charge Simulation Method and Its Applications,* Morikita, Tokyo, 1983 (in Japanese).
16. K. Murota, On "invariance" of schemes in the fundamental solution method, *Trans. Inform. Process, Soc. Japan* **34** (1993), 533–535 (in Japanese).
17. Z. Nehari, *Conformal Mapping,* McGraw-Hill, New York, 1952.
18. H. Okamoto and M. Katsurada, A rapid solver for the potential problems, *Bull. Japan SIAM* **2** (1992), 212–230 (in Japanese).
19. L. Reichel, A fast method for solving certain integral equations of the first kind with application to conformal mapping, *J. Comput. Appl. Math.* **14** (1986), 125–142.
20. H. Singer, H. Steinbigler and P. Weiss, A charge simulation method for the calculation of high voltage fields, *IEEE Trans. Power Apparatus Systems* **PAS-93** (1974), 1660–1668.
21. G.T. Symm, An integral equation method in conformal mapping, *Numer. Math.* **9** (1966), 250–258.
22. G.T. Symm, Numerical mapping of exterior domains, *Numer. Math.* **10** (1967), 437–445.
23. G.T. Symm, Conformal mapping of doubly-connected domains, *Numer. Math.* **13** (1969), 448–457.
24. L.N. Trefethen, Ed., *Numerical Conformal Mapping,* North-Holland, Amsterdam, 1986.

Keywords: numerical conformal mapping, multiply-connected domain, charge simulation method, method of fundamental solution, two-dimensional potential flow.

AMS subject classifications: 30C30, 65E05

Supported by the Grant-in-Aid for Scientific Research (B) 12440029 of Japan Society for the Promotion of Science.