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A Brief Overview of Fornberg-like Methods for Conformal Mapping of Simply and Multiply Connected Regions

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Dedicated to the memory of Mohamad Rashidi Md. Razali

Abstract. We give simplified derivations of Fornberg-like methods for conformal mapping of simply and multiply connected regions. The computational domains are circular domains and the derivations are based on Fourier series leading to conditions for analytic extension of boundary values to the computational domain. Linearization of these conditions with respect to the boundary correspondences and conformal moduli lead to Newton methods for approximating the mapping function. The linear systems can be solved by the conjugate gradient method.

1. Introduction

A great variety of numerical methods for efficiently computing conformal maps between various simply and multiply connected domains and conformally equivalent canonical domains have been developed in the past few decades. Thorough introductions to many of these methods can be found in the texts by Gaier [9] and Henrici [10] and a recent overview is given in a forthcoming survey by Wegmann [18]. Most general methods for numerical conformal mapping compute the boundary correspondences between the canonical and target domains and the conformal moduli. Canonical domains whose boundaries are circles are popular alternatives, since fast Fourier series methods can be used in the computation of the boundary correspondences. In this case, finding the conformal moduli amounts to calculating the centers and radii of conformally equivalent circular domains under certain normalizations of the mapping functions. Quadratically convergent Newton-like methods for solving these problems are particularly efficient. There are two main approaches to deriving such methods which have developed in parallel and use essentially the same linearization. One approach, due mainly to Wegmann, solves the inner linear systems for the Newton updates of the boundary correspondences and moduli as Riemann-Hilbert boundary value problems for the circular domains. The other approach, first proposed by Fornberg [7] for the simply connected case and extended by DeLillo et al., derives the inner linear systems from conditions on the Fourier (Laurent) coefficients which guarantee analytic extension of functions on the circles into the computational domain. Both approaches have been developed for the following canonical domains: the unit disk [7], [13], [6], the annulus [6], [12], [14], the ellipse [3], [4], [15], [11] and multiply-connected circular domains [5], [11], [1], [16], [17]. Iterative methods such as the conjugate gradient method can be used to solve the inner linear systems efficiently in most cases.

The purpose of this paper is to present a brief overview of the Fornberg-like methods. We will concentrate mainly on giving simplified derivations of the conditions on the Laurent coefficients for analytic extension and on the linearization. The details of the discretization and the numerical procedures and examples can be found in the references. Using FFTs and discretization by N-point trigonometric interpolation leads to computational costs of $O(N \log N)$ for the simply and doubly connected maps and $O(N^2)$ for the multiply connected maps. We discuss the disk map in Section 2, the annulus map in Section 3, the ellipse map in Section 4, and the multiply connected map in Section 5.

2. Conformal mapping of the unit disk

We wish to find the conformal map f from the interior of the unit disk D onto the interior Ω of a Jordan curve $\Gamma : \gamma(S), 0 \le S < L$. (The parameter S is often taken to be arclength). We assume here and below that the boundary curves γ are Hölder continuously differentiable with nonvanishing derivative and so contain no corners. The normalization imposed on f is

$$f(0) = 0, \quad f'(0) > 0 \quad \text{(or } f(1) = \gamma(0) \text{ fixed).} \tag{1}$$

Finding f is equivalent to finding the boundary correspondence function $S(\theta)$ where

$$f(e^{i\theta}) = \gamma(S(\theta)). \tag{2}$$

Now if D is the unit disk and f has the expansion

$$f(e^{i\theta}) := \sum_{k=-\infty}^{\infty} a_k \, e^{ik\theta} \tag{3}$$

then, by [[10], Sec. 14.3.I], f extends analytically into D if and only if

$$a_{-k} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, e^{ik\theta} d\theta = 0. \qquad k = 0, 1, 2, \cdots \tag{4}$$

That is, the negative-indexed Fourier coefficients of $f(e^{i\theta})$ must vanish. Note that the condition for $k = 0$ is obtained by applying the uniqueness condition $f(0) = 0$.

(For the exterior case [6], we fix $f(\infty) = \infty$ and the positive indexed coefficients of $f(z)/z$ must vanish.) We denote these conditions on the Fourier coefficients of f symbolically as

$$Pf = 0. \tag{5}$$

Here, for example, P can be factored as $P = I\_F$ where F is the Fourier transform, $Ff = (\cdots, a_{-1}, a_0, a_1, \cdots)^T$ are the Laurent coefficients, and $I_- = \text{diag}(\cdots, 1, 1, 0, 0, \cdots)$ zeros the positive–indexed coefficients, $Pf = I\_Ff = (\cdots, a_{-2}, a_{-1}, 0, 0, \cdots)^T$.

In order to compute a real correction $U^{(k)}(\theta)$ to the current approximation $S^{(k)}(\theta)$ to the exact $S(\theta)$, we linearize about $S^{(k)}(\theta)$ at the kth Newton step as follows:

$$\gamma(S^{(k)}(\theta) + U^{(k)}(\theta)) \approx \gamma(S^{(k)}(\theta)) + \gamma'(S^{(k)}(\theta))U^{(k)}(\theta). \tag{6}$$

Using $U^{(k)}(\theta)$ real and the analyticity conditions (5) gives a linear equation

$$P\gamma'(S^{(k)}(\theta))U^{(k)}(\theta) = -P\gamma(S^{(k)}(\theta)) \tag{7}$$

for the determination of $U^{(k)}$. In [6] it is shown how (7) can be converted to a second kind integral equation, discretized with the normalization added, and the resulting linear system solved efficiently with the conjugate gradient method. The Newton update is then given by

$$S^{(k+1)}(\theta) = S^{(k)}(\theta) + U^{(k)}(\theta) \to S(\theta). \tag{8}$$

3. Mapping of the annulus to a bounded doubly connected domain

In this section we will derive the Fornberg-like method for mapping an annulus D to a bounded, doubly connected region Ω as in [6]. Fornberg extended his method to doubly connected regions using a system of equations [[8], eq. (6)] that are closely related to the analyticity conditions derived in this section. He uses a linearly convergent method of successive approximation to solve this system. Here and in [6], we show how to linearize these equations to get a quadratically convergent Newton-like method.

According to a standard theorem, e.g. [[10], p. 445], for a given Ω there exists a unique real number ρ, $0 < \rho < 1$, such that the annulus $D : \rho < |z| < 1$ can be mapped conformally by a function f onto Ω. If the outer boundaries correspond to each other, then f is determined up to a rotation of the annulus. The number ρ is called the *conformal modulus* of the region Ω and is uniquely determined by Ω. Thus, determining ρ is part of the problem. The mapping f is uniquely determined by fixing a boundary point.

Let the target region Ω be bounded by two Jordan curves, the outer curve $\Gamma_1 : \gamma_1(S_1)$ and the inner curve $\Gamma_2 : \gamma_2(S_2)$. Our problem then is to find the boundary correspondences, $S_1(\theta)$ and $S_2(\theta)$, and the conformal modulus ρ, such that $f(z)$ is analytic in the annulus $D : \rho < |z| < 1$ with

$$f(e^{i\theta}) = \gamma_1(S_1(\theta)) \quad \text{and} \quad f(\rho e^{i\theta}) = \gamma_2(S_2(\theta)) \tag{9}$$

subject to the uniqueness condition $f(1) = \gamma_1(0)$.

If we have

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad f(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta} \tag{10}$$

then the necessary and sufficient conditions [6] for f to be analytic in D are

$$\rho^k a_k = b_k, \quad k = \cdots, -2, -1, 0, 1, 2, \cdots \tag{11}$$

(If $1 < |z| < \rho$, we have $\rho^k b_k = a_k$ instead.) We will denote these conditions symbolically as

$$P_1 f(e^{i\theta}) + P_2 f(\rho e^{i\theta}) = 0. \tag{12}$$

In order to compute corrections $U_1^{(k)}(\theta), U_2^{(k)}(\theta), \delta\rho^{(k)}$ to the current approximations $S_1^{(k)}(\theta)$, $S_2^{(k)}(\theta)$, $\rho^{(k)}$ of the exact $S_1(\theta), S_2(\theta), \rho$, we linearize about $S_1^{(k)}(\theta), S_2^{(k)}(\theta), \rho^{(k)}$ at the kth Newton step as follows:

$$\begin{aligned} f(e^{i\theta}) &= \gamma_1(S_1^{(k)}(\theta)) + \gamma_1'(S_1^{(k)}(\theta))U_1^{(k)}(\theta) \\ f(\rho^{(k)}e^{i\theta}) + f'(\rho^{(k)}e^{i\theta})\,\delta\rho^{(k)} &= \gamma_2(S_2^{(k)}(\theta) + \gamma_2'(S_2^{(k)}(\theta))U_2^{(k)}(\theta). \end{aligned} \tag{13}$$

This linearization was first proposed in [12]. Using the fact that $U_1^{(k)}, U_2^{(k)}, \delta\rho^{(k)}$ are real and the analyticity conditions (12), gives an equation

$$P_1\gamma_1'(S_1^{(k)})U_1^{(k)} + P_2\gamma_2'(S_2^{(k)})U_2^{(k)} - P_2 f'(\rho^{(k)}e^{i\theta})\delta\rho^{(k)} = -P_1\gamma_1(S_1^{(k)}) - P_2\gamma_2(S_2^{(k)}) \tag{14}$$

for the determination of $U_1^{(k)}, U_2^{(k)}, \delta\rho^{(k)}$. In [6] it is shown how (14) can be discretized and solved efficiently with conjugate gradient. The Newton update is then given by

$$\begin{aligned} S_1^{(k+1)}(\theta) &= S_1^{(k)}(\theta) + U_1^{(k)}(\theta) \to S_1(\theta), \\ S_2^{(k+1)}(\theta) &= S_2^{(k)}(\theta) + U_2^{(k)}(\theta) \to S_2(\theta), \\ \rho^{(k+1)} &= \rho^{(k)} + \delta\rho^{(k)} \to \rho. \end{aligned} \tag{15}$$

4. Mapping of an ellipse to an elongated region

One of the fundamental problems with the disk map is a severe ill-conditioning known as the crowding phenomonen; see [2] and references therein. The map from the disk to a region which is elongated in one direction has relative distortions which vary exponentially with the "aspect ratio" of the region, requiring many Fourier components for resolution. A better conditioned map can be found by using a similarly elongated ellipse as the canonical domain [3], [4], [15].

We now give a simplified derivation of the method based on the conditions for the annulus above; see [11]. This derivation is simpler than that in [3], [4] which was based on Chebyshev polynomials. Let $\Psi(\zeta) = \zeta + 1/\zeta$ be the familiar Joukowski map which maps $\rho e^{i\theta}$, $1 < \rho$, onto an ellipse $\Psi(\rho e^{i\theta})$. We want to find conditions on values of f on the boundary of the ellipse that guarantee analytic extension of f into the interior. Let

$$\begin{aligned} f(\Psi(\rho e^{i\theta})) &= \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \\ f(\Psi(e^{i\theta})) &= \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}. \end{aligned} \tag{16}$$

Note that $f(\Psi(z))$ maps the annulus $1 < |z| < \rho$ through the elliptical domain slit along $[-2, 2]$ onto the "slit" elongated region Ω. From Section 3 above, we have

$$\rho^k b_k = a_k, \quad k = \cdots, -2, -1, 0, 1, 2, \cdots. \tag{17}$$

In order to "close the slit", we force f to satisfy

$$f(\Psi(e^{i\theta})) = f(\Psi(e^{-i\theta})) \tag{18}$$

which leads to

$$b_k = b_{-k}. \tag{19}$$

Combining (17) and (19) gives us the desired analyticity conditions

$$a_k = \rho^{2k} a_{-k}, \quad k = 1, 2, \cdots. \tag{20}$$

We still have to incorporate normalization conditions into our linear system. We will discuss two choices for normalization: (i) $f(0) = 0$ and one boundary point fixed, and (ii) 3 boundary points fixed.

First, consider the condition $f(0) = 0$. Then

$$\begin{aligned} 0 = f(0) = f(\Psi(e^{i\pi/2})) &= \sum_{k=-\infty}^{\infty} a_k \rho^{-k} i^k \\ &= \cdots + a_{-3}\rho^3 i - a_{-2}\rho^2 - a_{-1}\rho i + a_0 + a_1\rho^{-1} i - a_2\rho^{-2} - a_3\rho^{-3} i + \cdots \end{aligned}$$

and

$$\begin{aligned} 0 = f(0) = f(\Psi(e^{-i\pi/2})) &= \sum_{k=-\infty}^{\infty} a_k \rho^{-k} (-i)^k \\ &= \cdots - a_{-3}\rho^3 i - a_{-2}\rho^2 + a_{-1}\rho i + a_0 - a_1\rho^{-1} i - a_2\rho^{-2} + a_3\rho^{-3} i + \cdots. \end{aligned}$$

Adding these two equations gives [[3], eq. (2.5)]

$$0 = a_0 + 2\sum_{k=1}^{\infty} (-1)^k \rho^{-2k} a_{2k}. \tag{21}$$

One boundary point can then be fixed, e.g., $f(1) = \gamma(0)$, by setting $S(0) = 0$.

Alternatively, three boundary points, say $f(\rho + 1/\rho)$, $f(i(\rho - 1/\rho))$, and $f(-\rho - 1/\rho)$, can be fixed by fixing $S(0)$, $S(\pi/2)$, $S(\pi)$.

The analyticity conditions, normalization, and linearization lead to equations similar to the simply connected case, eqs. (5), (6), (7), (8), above. Details are given in [3], [4]. In [4], a similar map is developed for the cross-shaped region using Faber polynomials.

5. Mapping of multiply connected circle domains

In this section, we will derive a simpler, symmetrized version of the method in [5], [11] for computing the conformal map f from an exterior n-connected circle domain to a domain exterior to n smooth Jordan curves. In [1], a derivation of this method is given following results in [5], [11] and further analysis and numerical examples are presented. Here, we give a simpler derivation based on series calculations suggested by Wegmann [19]. Methods for computing f based on Riemann-Hilbert problems are discussed in [[16], [17], [18]].

The conformal map f maps the complement, D, of n closed nonintersecting disks, $D_k, 1 \le k \le n$, onto a region Ω exterior to n smooth Jordan curves, $\Gamma_k, 1 \le k \le n$ with a simple pole at $\infty, f(\infty) = \infty$. The curves Γ_k and their interiors are nonintersecting. n is the *connectivity* of the Ω and D. We assume that $n \ge 2$. The circular disks D_k have boundaries C_k with centers z_k and radii $\rho_k, 1 \le k \le n$. The boundary of D is $C = C_1 + \cdots + C_n$, and the boundary of Ω is $\Gamma = \Gamma_1 + \cdots + \Gamma_n$, where $\Gamma_k : \gamma_k(S_k)$ is smooth (no corners) and $f(C_k) = \Gamma_k$. The parameter S_k need not be arclength.

According to the standard theorem, e.g., [[10], p. 488], given Ω, the circular domain D and the mapping function f, normalized by

$$f(z) = z + O(1/z) \tag{22}$$

near ∞, are uniquely determined. (In [5], the normalization $f(z) = Az + B + O(1/z)$ was used and A and B were determined by fixing four circle parameters.) The numerical problem is, therefore, to determine the boundary correspondence functions $S_k(\theta)$ and the centers z_k and radii r_k of the circles C_k, such that

$$f(z_k + \rho_k e^{ik\theta}) = \gamma_k(S_k(\theta)), \; k = 1, \cdots, n, \tag{23}$$

where f is analytic in D and satisfies (22).

From [[5], eq. (3)], the map f then has the representation

$$f(z) = z + \sum_{k=1}^{n} H_k(z) \tag{24}$$

with

$$H_k(z) = \sum_{j=1}^{\infty} b_{k,-j} \left(\frac{\rho_k}{z - z_k} \right)^j. \tag{25}$$

Consider the Fourier series from [[5], Theorem 5],

$$f(z_k + \rho_k e^{i\theta}) = \sum_{j=-\infty}^{\infty} a_{kj} e^{ij\theta}. \tag{26}$$

Inserting (24) into (26), we have

$$f(z_k + \rho_k e^{i\theta}) = z_k + \rho_k e^{i\theta} + \sum_{j=1}^{\infty} b_{k,-j} e^{ij\theta} + \sum_{l \ne k} H_l(z_k + \rho_k e^{i\theta}) \tag{27}$$

Since the functions H_l for $l \neq k$ are analytic in D_k the last term in (27) contributes only Fourier-terms $e^{ij\theta}$ with $j \geq 0$. Comparison of coefficients in (26) and (27) yields

$$b_{kj} = a_{kj} \quad \text{for} \quad j < 0. \tag{28}$$

Comparison of the remaining coefficients in (25) and (26) gives our analyticity conditions (30), as follows. Using the binomial series

$$(z - z_l)^{-j} = (z_k - z_l)^{-j} \sum_{m=0}^{\infty} \binom{-j}{m} \left(\frac{z - z_k}{z_k - z_l} \right)^m \tag{29}$$

one can calculate, from the Laurent series (25) of H_l, the Taylor series of H_l at the point z_k. Comparison of coefficients in (27) using (28) yields

$$a_{kj} = \sum_{l \neq k} \sum_{m=1}^{\infty} \binom{-m}{j} a_{l,-m} \, \rho_l^m \rho_k^j \left(z_k - z_l\right)^{-m-j} \tag{30}$$

for $j \geq 2$. For $j = 0$ and $j = 1$ one has to add z_k and ρ_k, respectively, to the right hand side of (30).

In [1], it is shown that (30) are just a symmetrized version of the conditions in [[5], Theorem 5] where the condition for $k = 1$ is used for each k with z_k replacing z_1. This can be seen by noting that, for $B_{m,j}$ in [5], we have

$$\binom{-m}{j} = (-1)^j \frac{(m + j - 1)!}{j!(m-1)!} = (-1)^j B_{m,j} \quad \text{and} \quad B_{j+1,m-1} = B_{m,j}. \tag{31}$$

Hence, (30) can be rewritten as

$$x_j = a_{k,j} - \sum_{l \neq k} \left(\frac{\rho_1}{z_l - z_k} \right)^j \sum_{\nu=0}^{\infty} B_{j+1,\nu} \left(\frac{\rho_2}{z_k - z_l} \right)^{\nu+1} a_{l,-\nu-1} \tag{32}$$

where $x_0 = z_k, x_1 = \rho_k$, and $x_j = 0$ for $j \geq 2$. We denote these conditions symbolically as

$$Pf = \sum_{k=1}^{n} P_k f_k = r \tag{33}$$

where $f_k = f(z_k + \rho_k e^{i\theta})$.

As in [5], it is now necessary to also linearize about the z_k's

$$f\left(z_k + \delta z_k + (\rho_k + \delta\rho_k)e^{i\theta}\right) \approx f\left(z_k + \rho_k e^{i\theta}\right) + f'\left(z_k + \rho_k e^{i\theta}\right)\left(\delta z_k + \delta\rho_k e^{i\theta}\right) \tag{34}$$

giving

$$f\left(z_k^{(m)} + \rho_k^{(m)} e^{i\theta}\right) + f'\left(z_k^{(m)} + \rho_k^{(m)} e^{i\theta}\right)\left(\delta z_k^{(m)} + \delta\rho_k^{(m)} e^{i\theta}\right) = \gamma_k\left(S_k^{(m)}(\theta)\right) + \gamma_k'\left(S_k^{(m)}(\theta)\right) U_k^{(m)}(\theta) \tag{35}$$

for the mth Newton step. This linearization is rigorously justified in [16].

Representing the unknown updates U_k, ρ_k, real(z_k), imag(z_k), $k = 1, \cdots, n$ by U, we can derive a linear system for U which we denote symbolically as

$$PU = g. \tag{36}$$

The details of the discretization, setup, and solution of this system by the conjugate gradient method are given in [1] and are similar to [5]. We will not present them here. The mth Newton updates are

$$\begin{aligned} S_k^{(m+1)}(\theta) &= S_k^{(m)}(\theta) + U_k^{(m)}(\theta) \to S_k(\theta), \\ \rho_k^{(m+1)} &= \rho_k^{(m)} + \delta\rho_k^{(m)} \to \rho_k, \\ z_k^{(m+1)} &= z_k^{(m)} + \delta z_k^{(m)} \to z_k \end{aligned} \tag{37}$$

for $k = 1, \cdots, n$, similar to (15).

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