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Coefficients of the Inverse of Strongly Starlike Functions

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Dedicated to the memory of Professor Mohamad Rashidi Md. Razali

Abstract. For the class of strongly starlike functions, sharp bounds on the first four coefficients of the inverse functions are determined. A sharp estimate for the Fekete-Szegö coefficient functional is also obtained. These results were deduced from non-linear coefficient estimates of functions with positive real part.

1. Introduction

An analytic function f in the open unit disk $U = \{z : |z| < 1\}$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if f is normalized by $f(0) = 0 = f'(0) - 1$ and satisfies

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi\alpha}{2} \quad (z \in U).$$

The set of all such functions is denoted by $SS^*(\alpha)$. This class has been studied by several authors $[1, 2, 5, 7, 9, 10]$. In [5] it was shown that an univalent function f belongs to $SS^*(\alpha)$ if and only if for every $w \in f(U)$ a certain lens-shaped region with vertices at the origin O and w is contained in $f(U)$.

If

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

is in the class $SS^*(\alpha)$, then the inverse of f admits an expansion

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots \tag{2}$$

near $w = 0$. It is our purpose here to determine sharp bounds for the first four coefficients of $|\gamma_n|$, and to obtain a sharp estimate for the Fekete-Szegö coefficient functional $\left|\gamma_3 - t\gamma_2^2\right|$.

2. Preliminary results

Let us denote by P the class of normalized analytic functions p in the unit disk U with positive real part so that $p(0) = 1$ and $\operatorname{Re} p(z) > 0,\ z \in U$. It is clear that $f \in SS^*(\alpha)$ if and only if there exists a function $p \in P$ so that $zf'(z)/f(z) = p^{\alpha}(z)$. By equating coefficients, each coefficient of $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ can be expressed in terms of coefficients of a function $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ in the class P. For example,

$$\begin{aligned} a_2 &= \alpha c_1 \\ a_3 &= \frac{\alpha}{2}\left[c_2 - \frac{1-3\alpha}{2} c_1^2\right] \\ a_4 &= \frac{\alpha}{3}\left[c_3 + \frac{5\alpha - 2}{2} c_1 c_2 + \frac{17\alpha^2 - 15\alpha + 4}{12} c_1^3\right] \end{aligned} \tag{3}$$

Using representations (1) and (2) together with $f(f^{-1}(w)) = w$ or

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots,$$

we obtain the relationships

$$\begin{aligned} \gamma_2 &= -a_2 \\ \gamma_3 &= -a_3 + 2a_2^2 \\ \gamma_4 &= -a_4 + 5a_2 a_3 - 5a_2^3 \end{aligned} \tag{4}$$

Thus coefficient estimates for the class $SS^*(\alpha)$ and its inverses become non-linear coefficient problems for the class P. Our principal tool is given in the following lemma.

Lemma 1 [3]. *A function* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *belongs to* P *if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| 2z_j + \sum_{k=1}^{\infty} c_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} c_{k+1} z_{k+j} \right|^2 \right\} \geq 0$$

for every sequence $\{z_k\}$ of complex numbers which satisfy $\lim_{k\to\infty} \sup |z_k|^{1/k} < 1$.

Lemma 2. *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, *then*

$$\left| c_2 - \frac{\mu}{2} c_1^2 \right| \leq \max\{2, 2|\mu - 1|\} = \begin{cases} 2 & , \quad 0 \leq \mu \leq 2 \\ 2\,|\,\mu - 1\,| & , \quad \textit{elsewhere} \end{cases}$$

If $\mu < 0$ *or* $\mu > 2$, *equality holds if and only if* $p(z) = (1 + \varepsilon z)/(1 - \varepsilon z)$, $|\varepsilon| = 1$. *If* $0 < \mu < 2$, *then equality holds if and only if* $p(z) = (1 + \varepsilon z^2)/(1 - \varepsilon z^2)$, $|\varepsilon| = 1$. *For* $\mu = 0$, *equality holds if and only if*

$$p(z) := p_2(z) = \lambda \frac{1 + \varepsilon z}{1 - \varepsilon z} + (1 - \lambda) \frac{1 - \varepsilon z}{1 + \varepsilon z}, \quad 0 \leq \lambda \leq 1, \ |\varepsilon| = 1.$$

For $\mu = 2$, *equality holds if and only if* p *is the reciprocal of* p_2.

Remark. Ma and Minda [6] had earlier proved the above result. We give a different proof.

Proof. Choose the sequence $\{z_k\}$ of complex numbers in Lemma 1 to be $z_0 = -\mu c_1/2$, $z_1 = 1$, and $z_k = 0$ if $k > 1$. This yields

$$\left| c_2 - \frac{\mu}{2} c_1^2 \right|^2 + |c_1|^2 \leq |(1 - \mu) c_1|^2 + 4,$$

that is,

$$\left| c_2 - \frac{\mu}{2} c_1^2 \right|^2 \leq 4 + \mu(\mu - 2)|c_1|^2. \tag{5}$$

If $\mu < 0$ or $\mu > 2$, the expression on the right of inequality (5) is bounded above by $4(\mu - 1)^2$. Equality holds if and only if $|c_1| = 2$, i.e., $p(z) = (1+z)/(1-z)$ or its rotations. If $0 < \mu < 2$, then the right expression of inequality (5) is bounded above by 4. In this case, equality holds if and only if $|c_1| = 0$ and $|c_2| = 2$, i.e., $p(z) = (1+z^2)/(1-z^2)$ or its rotations. Equality holds when $\mu = 0$ if and only if $|c_2| = 2$, i.e., [8, p. 41]

$$p(z) := p_2(z) = \lambda \frac{1+\varepsilon z}{1-\varepsilon z} + (1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}, \quad 0 \le \lambda \le 1, \; |\varepsilon| = 1.$$

Finally, when $\mu = 2$, then $|c_2 - c_1^2| = 2$ if and only if p is the reciprocal of p_2.

Another interesting application of Lemma 1 occurs by choosing the sequence $\{z_k\}$ to be $z_0 = \delta c_1^2 - \beta c_2$, $z_1 = -\gamma c_1$, $z_2 = 1$, and $z_k = 0$ if $k > 2$. In this case, we find that

$$\begin{aligned} \left|c_3 - (\beta+\gamma)c_1c_2 + \delta c_1^3\right|^2 &\le 4 + 4\gamma(\gamma-1)|c_1|^2 + \left|(2\delta-\gamma)c_1^2 - (2\beta-1)c_2\right|^2 \\ &\quad - \left|c_2 - \gamma c_1^2\right|^2 = 4 + 4\gamma(\gamma-1)|c_1|^2 + 4\beta(\beta-1)\left|c_2 - \frac{\nu}{2}c_1^2\right|^2 \\ &\quad - \frac{(\delta-\beta\gamma)^2}{\beta(\beta-1)}|c_1|^4 \end{aligned} \tag{6}$$

where $\nu := \dfrac{\delta(\beta-1) + \beta(\delta-\gamma)}{\beta(\beta-1)}$.

Lemma 3. *Let* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$. *If* $0 \le \beta \le 1$ *and* $\beta(2\beta-1) \le \delta \le \beta$, *then*

$$\left|c_3 - 2\beta c_1 c_2 + \delta c_1^3\right| \le 2.$$

Proof. If $\beta = 0$, then $\delta = 0$ and the result follows since $|c_3| \le 2$. If $\beta = 1$, then $\delta = 1$ and the inequality follows from a result of [4].

We may assume that $0 < \beta < 1$ so that $\beta(\beta-1) < 0$. With $\gamma = \beta$, we find from (6) that

$$\left| c_3 - 2\beta c_1 c_2 + \delta c_1^3 \right|^2 \le 4 + 4\beta(\beta - 1)\,|c_1|^2 + 4\beta(\beta - 1)\left| c_2 - \frac{\nu}{2} c_1^2 \right|^2$$

$$- \frac{(\delta - \beta^2)^2}{\beta(\beta - 1)}\,|c_1|^4 \le 4 + bx + cx^2 := h(x)$$

with $x = |c_1|^2 \in [0, 4]$, $b = 4\beta(\beta - 1)$, and $c = -(\delta - \beta^2)^2 / \beta(\beta - 1)$. Since $c \ge 0$, it follows that $h(x) \le h(0)$ provided $h(0) - h(4) \ge 0$, i.e., $b + 4c \le 0$. This condition is equivalent to $|\delta - \beta^2| \le \beta(1 - \beta)$, which completes the proof.

With $\delta = \beta$ in Lemma 3, we obtain an extension of Libera and Zlotkiewicz [4] result that $\left| c_3 - 2c_1 c_2 + c_1^3 \right| \le 2$.

Corollary 1. *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, *and* $0 \le \beta \le 1$, *then*

$$\left| c_3 - 2\beta c_1 c_2 + \beta c_1^3 \right| \le 2.$$

When $\beta = 0$, *equality holds if and only if*

$$p(z) := p_3(z) = \sum_{k=1}^{3} \lambda_k \frac{1 + \varepsilon e^{-2\pi ik/3} z}{1 - \varepsilon e^{-2\pi ik/3} z}, \quad (|\varepsilon| = 1)$$

$\lambda_k \ge 0$, *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *If* $\beta = 1$, *equality holds if and only if* p *is the reciprocal of* p_3. *If* $0 < \beta < 1$, *equality holds if and only if* $p(z) = (1 + \varepsilon z)/1 - \varepsilon z)$, $|\varepsilon| = 1$, *or* $p(z) = (1 + \varepsilon z^3)/(1 - \varepsilon z^3)$, $|\varepsilon| = 1$.

Proof. We only need to find the extremal functions. If $\beta = 0$, then equality holds if and only if $|c_3| = 2$, i.e., p is the function p_3 [8, p. 41]. If $\beta = 1$, then equality holds if and only if p is the reciprocal of p_3. When $0 < \beta < 1$, we deduce from (6) that

$$\left| c_3 - 2\beta c_1 c_2 + \beta c_1^3 \right|^2 \le 4 + 4\beta(\beta - 1)\,|c_1|^2 + 4\beta(\beta - 1)\left| c_2 - \frac{1}{2} c_1^2 \right|^2$$

$$- \beta(\beta - 1)\,|c_1|^4 \le 4 + 4\beta(\beta - 1)\,|c_1|^2 - \beta(\beta - 1)\,|c_1|^4 \le 4.$$

Equality occurs in the last inequality if and only if either $|c_1| = 0$ or $|c_1| = 2$. If $|c_1| = 0$, then $|c_2| = 0$, i.e., $p(z) = (1+\varepsilon z^3)\big/(1-\varepsilon z^3)$, $|\varepsilon| = 1$. If $|c_1| = 2$, then $p(z) = (1+\varepsilon z)\big/(1-\varepsilon z)$, $|\varepsilon| = 1$.

Lemma 4. *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, *then*

$$\left|c_3 - (\mu+1)c_1c_2 + \mu c_1^3\right| \le \max\left\{2, 2\left|2\mu - 1\right|\right\} = \begin{cases} 2 & , \quad 0 \le \mu \le 1 \\ 2\left|2\mu - 1\right| & , \quad \textit{elsewhere} \end{cases}$$

Proof. For $0 \le \mu \le 1$, the inequality follows from Lemma 3 with $\delta = \mu$, and $2\beta = \mu + 1$. For the second estimate, choose $\beta = \mu$, $\gamma = 1$, and $\delta = \mu$ in (6). Since $\mu(\mu - 1) > 0$, we conclude from (5) and (6) that

$$\left| c_3 - (\mu+1)c_1c_2 + \mu c_1^3\right|^2 \le 4 + 4\mu(\mu - 1)\left| c_2 - c_1^2\right|^2 \le 4(2\mu - 1)^2.$$

3. Coefficient bounds

Theorem 1. *Let* $f \in SS^*(\alpha)$ *and* $f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$. *Then*

$$|\gamma_2| \le 2\alpha,$$

with equality if and only if

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\varepsilon z}{1-\varepsilon z}\right)^{\alpha}, \quad |\varepsilon| = 1. \tag{7}$$

Further

$$|\gamma_3| \le \begin{cases} \alpha & , \quad 0 < \alpha \le \frac{1}{5} \\ 5\alpha^2 & , \quad \frac{1}{5} \le \alpha \le 1 \end{cases}$$

For $\alpha > 1/5$, *extremal functions are given by* (7). *If* $0 < \alpha < 1/5$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\varepsilon z^2}{1-\varepsilon z^2}\right)^{\alpha}, \quad |\varepsilon| = 1, \tag{8}$$

while if $\alpha = 1/5$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)} = p_2(z)^{-\alpha} = \left(\lambda\frac{1+\varepsilon z}{1-\varepsilon z} + (1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}\right)^{-\alpha}, \; |\varepsilon| = 1, \; 0 \le \lambda \le 1.$$

Moreover,

$$|\gamma_4| \le \begin{cases} \dfrac{2\alpha}{3} & , \; 0 < \alpha \le \dfrac{1}{\sqrt{31}} \\ \dfrac{2\alpha}{9}\left(62\alpha^2 + 1\right) & , \; \dfrac{1}{\sqrt{3}} \le \alpha \le 1 \end{cases}$$

For $\alpha \ge 1/\sqrt{31}$, *extremal functions are given by* (7), *while for* $0 < \alpha \le 1/\sqrt{31}$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\varepsilon z^3}{1-\varepsilon z^3}\right)^{\alpha}, \; |\varepsilon| = 1.$$

Proof. The following relations are obtained from (3) and (4):

$$\begin{aligned} \gamma_2 &= -\alpha c_1 \\ \gamma_3 &= -\frac{\alpha}{2}\left[c_2 - \frac{1+5\alpha}{2}c_1{}^2\right] \\ \gamma_4 &= -\frac{\alpha}{3}\left[c_3 - (1+5\alpha)c_1c_2 + \frac{31\alpha^2 + 15\alpha + 2}{6}c_1{}^3\right] := -\frac{\alpha}{3}E \end{aligned} \tag{9}$$

The bound on $|\gamma_2|$ follows immediately from the well-known inequality $|c_1| \le 2.$ Lemma 2 with $\mu = 1 + 5\alpha$ yields the bound on $|\gamma_3|$ and the description of the extremal functions.

For the fourth coefficient, we shall apply Lemma 3 with $2\beta = 1 + 5\alpha$ and $\delta = (31\alpha^2 + 15\alpha + 2)/6.$ The conditions on β and δ are satisfied if $\alpha \le 1/\sqrt{31}$. Thus $|\gamma_4| \le 2\alpha/3$, with equality if and only if $zf'(z)/f(z) = \left[\left(1+\varepsilon z^3\right)/\left(1-\varepsilon z^3\right)\right]^{\alpha}$.

For $1/\sqrt{31} < \alpha \le 1/5$, Corollary 1 yields

$$|E| \le \left|c_3 - (1+5\alpha)c_1c_2 + \frac{1+5\alpha}{2}c_1{}^3\right| + \frac{31\alpha^2 - 1}{6}|c_1|^3 \le \frac{2}{3}\left(62\alpha^2 + 1\right).$$

It remains to determine the estimate for $1/5 < \alpha \le 1$. Appealing to Lemma 4 with $\mu = 5\alpha$, and because $31\alpha^2 - 15\alpha + 2 > 0$ in $(0,1]$, we conclude that

$$|E| \le \left| c_3 - (1+5\alpha)c_1c_2 + 5\alpha c_1^3 \right| + \frac{31\alpha^2 - 15\alpha + 2}{6}|c_1|^3 \le 2(10\alpha - 1) + \frac{4}{3}(31\alpha^2 - 15\alpha + 2) = \frac{2}{3}\left(62\alpha^2 + 1\right).$$

Theorem 2. *Let* $f \in SS^*(\alpha)$ *and* $f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$. *Then*

$$\left|\gamma_3 - t\gamma_2^2\right| \le \begin{cases} (5-4t)\alpha^2 & , \quad t \le \dfrac{5 - 1/\alpha}{4} \\ \alpha & , \quad \dfrac{5-1/\alpha}{4} \le t \le \dfrac{5+1/\alpha}{4} \\ (4t-5)\alpha^2 & , \quad t \ge \dfrac{5+1/\alpha}{4} \end{cases}$$

If $\frac{5-1/\alpha}{4} < t < \frac{5+1/\alpha}{4}$, *equality holds if and only if* f *is given by* (8). *If* $t < \frac{5-1/\alpha}{4}$ *or* $t > \frac{5+1/\alpha}{4}$, *equality holds if and only if* f *is given by* (7). *If* $t = \frac{5+1/\alpha}{4}$, *equality holds if and only if* $\frac{zf'(z)}{f(z)} = p_2(z)^\alpha$, *while if* $t = \frac{5-1/\alpha}{4}$, *then equality holds if and only if* $\frac{zf'(z)}{f(z)} = p_2(z)^{-\alpha}$.

Proof. From (9), we obtain

$$\gamma_3 - t\gamma_2^2 = -\frac{\alpha}{2}\left[c_2 - \frac{1+(5-4t)\alpha}{2}c_1^2\right].$$

The result now follows from Lemma 2.

Remark. An equivalent result for the Fekete-Szegö coefficient functional over the class $SS^*(\alpha)$ was also given by Ma and Minda [5].

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