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On sM-Group

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Abstract: A subnormally monomial group (abbreviated sM-group) is a finite group all whose irreducible complex characters are induced from linear characters of some subnormal subgroups. It has been conjectured that the derived length of such groups is bounded by a constant. This paper gives a negative answer. An sM-group of derived length 5 is constructed and we believe that the same method can be used to construct sM-group of arbitrary derived length.

1. Introduction

The study of monomial groups is stimulated by the study of Artin's *L*-function arising from number theory. Artin defined the *L*-function in 1923 (see [2]). Roughly speaking, the *L*-function $L(s, \chi, K/k)$ is a function of a complex variable s, and depends on a complex character χ of the Galois group of a Galois extension K/k of an algebraic number field k of finite degree. In general, *L* is meromorphic in the whole complex plane. Artin conjectured that L is an entire function. One positive answer known concerns the case when χ is a monomial character. G.A. How ([3]) investigated a special class of monomial group, namely the sM-group. It was conjectured that the derived length of an sM-group is bounded. This paper constructed an sM-group of derived length 5 and it indicates that the same method can be used to construct sM-group of arbitrary derived length.

2. Definitions and some theorems

All groups considered are finite and all characters are over the complex number field.

Definition 2.1. *An irreducible character of a group G is called subnormally monomial iff it is induced from a linear character of some subnormal subgroup. G is a subnormally monomial group (abbreviated sM-group) iff all its irreducible characters are subnormally monomial.*

G.A.How [3] characterized sM-groups which are *A*-groups as chiefly sub-Frobenius groups whose definition is given below.

Definition 2.2. *A solvable group is called chiefly sub-Frobenius if* $C_G(kL)$ *is subnormal in G whenever* kL *is an element of a chief factor* K/L *of G.*

Theorem 2.3. *Given that G is an A-group. Then G is an sM-group iff G is chiefly sub-Frobenius.*

Proof. Refer to [3].

Let $\sigma(G)$ denote the socle of *G*, i.e. the subgroup generated by all the minimal normal subgroups of *G*. A few well-known properties of socle are listed below (see [5]).

Theorem 2.4.
(1) $\sigma(G)$ *is characteristic in G.*
(2) $\sigma(G)$ *is abelian.*
(3) $\sigma(G)$ *is the direct product of some of the minimal normal subgroups of G.*
(4) $\sigma(G \times H) = \sigma(G) \times \sigma(H)$.
(5) For each normal subgroup N of G contained in $\sigma(G)$, *there is a normal subgroup M in G such that* $\sigma(G) = N \times M$.

Theorem 2.5. *If* $C_G(\sigma(G)) = \sigma(G)$, *and N is a maximal normal subgroup of G, then*
(1) $C_N(\sigma(N)) = \sigma(N)$,
(2) $\sigma(N) = N \cap \sigma(G)$,
(3) $\sigma(G) = (N \cap \sigma(G)) \times L$ *where L is central in G.*

Proof. Consider first the case $\sigma(G) \le N$. Then (3) is trivially true with $L = \{1\}$. As $\sigma(N)$ is an abelian normal subgroup of *G*, it will contain or avoid, and hence centralize, each minimal normal subgroup of G. Thus $\sigma(N)$ centralizes $\sigma(G)$, and by assumption, we have $\sigma(N) \le \sigma(G)$. By part (5) of Theorem 2.4, $\sigma(G) = \sigma(N) \times M$ where *M* is some normal subgroup of *G*. However, we have now *M* is normal in *N* and avoids $\sigma(N)$, this forces $M = \{1\}$ and $\sigma(N) = \sigma(G)$. Thus (1) and (2) follow immediately.

Next we suppose $\sigma(G)$ is not inside *N*, and let *L* be a minimal normal subgroup of *G* that is not contained in *N*. Since *N* is a maximal normal subgroup, we have $G = N \times L$ and *L* is central in *G*. By part (4) of Theorem 2.4, $\sigma(G) = \sigma(N) \times L$ and so (2) and (3) hold. As *L* is central, $C_G(\sigma(N)) = C_G(\sigma(G)) = \sigma(G)$. Thus $C_N(\sigma(N)) = N \cap C_G(\sigma(G)) = \sigma(N)$.

Theorem 2.6. *If* $C_G(\sigma(G)) = \sigma(G)$, *and R is a subnormal subgroup of G, then R is a direct factor of* $R\sigma(G)$.

Proof. We use induction on $|G|$. If $R = G$, there is nothing to prove, so we may assume that R is contained in some maximal normal subgroup N of G. By part (1) of Theorem 2.5, the inductive hypothesis applies to N, and we conclude that $R\sigma(N) = R \times K$ for some K. By part (2) and (3) of Theorem 2.5, $\sigma(G) = (N \cap \sigma(G)) \times L = \sigma(N) \times L$, with L central in G. Thus, $R\sigma(G) = R(\sigma(N) \times L) = R\sigma(N) \times L$ (note that $R\sigma(N) \cap L \le N \cap L = \{1\}$), and that $R\sigma(G) = (R \times K) \times L = R \times (K \times L)$. The proof is complete.

3. sM-group with derived length 5

The group we are going to construct is an A-group of derived length 5. To show that this group is an sM-group, we need only to prove that it is chiefly sub-Frobenius.

We start with the cyclic groups $\langle a : a^{16} = 1\rangle$ and $\langle b : b^{27} = 1\rangle$. Let a act on b invertingly, and form the corresponding split extension

$$G = \langle a, b : a^{16} = b^{27} = 1,\ b^a = b^{-1}\rangle$$

Note that $G' = \langle b\rangle$, and $\langle a^2 b\rangle$ is a cyclic subgroup of index 2; in particular, G is certainly a chiefly sub-Frobenius group. We shall need repeatedly the following fact. If $g \in G,\ i \ge 1$, and $a^{2^i} \notin \langle g\rangle,\ b^{3^{i-1}} \notin \langle g\rangle$, then $g \in \langle a^{2^{i+1}} b^{3^i}\rangle$.

Let G act on $\langle c, d : c^{49} = d^{49} = 1,\ cd = dc\rangle$ so that $c^a = d,\ d^a = c^{-1},\ c^b = c^{18},\ d^b = d^{30}$. As $18^3 \equiv 18 \times 30 \equiv 1 \pmod{49}$, it is straightforward to check that this definition is legitimate. One may also view $\langle c, d\rangle$ as a G-module induced from the $\langle a^2 b\rangle$-module $\langle c\rangle$ such that $c^{a^2 b} = c^{-18}$. As a^2 and b act fixed point free but $a^4 b^3$ acts trivially on $\langle c, d\rangle$, the comment above with $i = 1$ makes it easy to check that $H = \langle c, d\rangle$ split G is a chiefly sub-Frobenius group. Also, $H' = \langle b, c, d\rangle$ and $H'' = \langle c, d\rangle$. The Fitting subgroup $F(H)$ is $\langle a^4, b^3, c, d\rangle$, and $F(H)/\langle a^8, b^9, c^7, d\rangle$ is cyclic of order 42.

Consider a 1-dimensional faithful $F(H)/\langle a^8, b^9, c^7, d\rangle$-module U over the field of order 43, as an $F(H)$-module, and form the induced H-module U^H. By Mackey's Subgroup Theorem, $(U^H)_{F(H)} = \oplus_{i=1}^{12} U_i$ where the U_i are the conjugates of U under a set of representatives of the cosets of $F(H)$ in H. Let the numbering be arranged so that $U_1 = U$ and $U_1, \cdots, U_6$ are the conjugates under $\langle a^2, b, c, d\rangle$; then $\langle a^2, b, c, d\rangle$ normalizes and a interchanges $\oplus_{i=1}^{6} U_i$ and $\oplus_{i=7}^{12} U_i$. As $\langle c\rangle$ and $\langle d\rangle$ are also normalized by $\langle a^2, b, c, d\rangle$ and interchanged by a, we readily see that c acts fixed point freely on the first sum and trivially on the second, while d acts trivially on the first and fixed point freely on the second. Of course $\langle a^8, b^9, c^7, d^7\rangle$ acts trivially while a^4 and b^3 act fixed point freely on both. In particular, if $h \in \langle c, d\rangle$, then

$$[U^H, h] = \begin{cases} 1 & , \quad h \in \langle c^7, d^7\rangle \\ \oplus_{i=1}^{6} U_i & , \quad h \in \langle c, d^7\rangle - \langle c^7, d^7\rangle \\ \oplus_{i=7}^{12} U_i & , \quad h \in \langle c^7, d\rangle - \langle c^7, d^7\rangle \\ U^H & , \quad \text{otherwise} \end{cases}$$

Now form $K = U^H$ split H. Then $K' = U^H H'$ and $K'' = U^H H'' = U^H \langle c, d\rangle$, so $K''' = U^H$. To verify that K is a chiefly sub-Frobenius group, we need only established that if $0 \neq u \in U^H$, then $C_H(u)$ is subnormal in H. Since $C_H(u)$ is generated by its elements x of prime power order, and since joins of subnormal subgroups are subnormal (refer to [7]), this will be proved if we show that the subnormal closure of each such x centralizes u. When x is a 7-element, it lies in the normal Sylow 7-subgroup $\langle c, d\rangle$ of H, so $\langle x\rangle$ itself is subnormal and there is no work to do. When x is a 2-element or a 3-element, then some conjugate x^y lies in the Hall $\{2,3\}$-subgroup G of H. As this x^y fixes the nonzero u^y, we have $a^4 \notin \langle x^y\rangle$ and $b^3 \notin \langle x^y\rangle$, so $x^y \in \langle a^8 b^9\rangle$. Thus if x^y is a 2-element, it is central in H, while if it is a 3-element, it is central in the normal subgroup $\langle b, c, d\rangle$ of H; so $\langle x^y\rangle$ is subnormal. Consequently, so is $\langle x\rangle$, and we are done.

We shall need later on that if $h \in \langle c, d \rangle$ then the subnormal closure S of h in K is $[U^H, h]\langle h \rangle$. To see this, we observe that $U^H \leq \sigma(K)$, and recall that by Theorem 1.6 S is a direct factor of $S\sigma(K)$. Therefore S is a direct factor of SU^H, and hence $[U^H, h] \leq [SU^H, S] \leq S$. Conversely, $[U^H, h]\langle h \rangle$ is normal in $U^H \langle h \rangle$ which is subnormal in K, so $S \leq [U^H, h]\langle h \rangle$.

Note that the Fitting subgroup $F(K)$ is $U^H \langle a^8, b^9, c^7, d^7 \rangle$, of index 3528 in K. Choose a prime p such that $p \equiv 1 \bmod (2 \times 3 \times 7 \times 43)$ (this is possible by Dirichlet 's Theorem), and a faithful 1-dimensional module V for the cyclic group $F(K) \big/ \left(\oplus_{i=2}^{11} U_i\right)\langle d^7 \rangle$ of order $2 \times 3 \times 7 \times 43$ over the field of order p. Regard V as an $F(K)$-module, and consider the induced module V^K. By Mackey's Subgroup Theorem, $(V^K)_{F(K)}$ is the direct sum of 3528 conjugates V_i of V. Number these so that $V_1 = V$ and the 1764 conjugates under $U^H \langle a^2, b, c, d \rangle$ are listed first. Let W_1 be the sum of these and W_2 be the sum of the other V_i. Then $U^H \langle a^2, b, c, d \rangle$ normalizes and a interchanges W_1 and W_2, and also $\langle c^7 \rangle$ and $\langle d^7 \rangle$. It follows that c^7 acts fixed point freely on W_1 and trivially on W_2, while d^7 acts trivially on W_1 and fixed point freely on W_2. Similarly, $\oplus_{i=7}^{12} U_i$ acts trivially on W_1 and $\oplus_{i=1}^{6} U_i$ acts trivially on W_2.

The proof of the fact that V^K split K is a chiefly sub-Frobenius group follows the previous pattern. The only step that is different is to show that if $0 \neq v \in V^K$ and $h \in \langle c, d \rangle \cap C_K(v)$, then the subnormal closure of h in K centralizes v. If $h \in F(K)$, then $\langle h \rangle$ is subnormal and there is nothing to prove. If $h \notin F(K)$, i.e. $h \notin \langle c^7, d^7 \rangle$ then h^7 is nontrivial and fixes the nonzero v, hence $h^7 \in \langle c^7 \rangle$ or $\langle d^7 \rangle$ (otherwise it would act fixed point freely on both W_1 and W_2). Say $\langle h^7 \rangle = \langle c^7 \rangle$; then $v \in W_2$. Also, $h \in \langle c, d^7 \rangle - \langle c^7, d^7 \rangle$, so $S = [U^H, h]\langle h \rangle = \left(\oplus_{i=1}^{6} U_i\right)\langle h \rangle$, and as $\oplus_{i=1}^{6} U_i$ acts trivially on W_2, S does centralizes v.

It remains to note that $K''' = U^H$ acts nontrivially on V^K and therefore K''' cannot lie in the Fitting subgroup of V^K split K. Hence $(V^K K)'''$ is nonabelian, and therefore the derived length of $V^K K$ is 5.

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