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On the Chromatic Uniqueness of Edge-Gluing of Complete Tripartite Graphs and Cycles

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Abstract. In this paper, it is shown that the graph obtained by overlapping the cycle $C_m\,(m \geq 3)$ and the complete tripartite graph $K_{2,2,2}$ at an edge is uniquely determined by its chromatic polynomial.

Let G be a finite graph with neither loops nor multiple edges and let $P(G;\lambda)$ denote its chromatic polynomial. Then G is said to be *chromatically unique* if $P(Y;\lambda) = (G;\lambda)$ implies that Y is isomorphic to G.

Let K_n and C_n denote a complete graph and a cycle respectively on n vertices. The complete t-partite graph whose t partite sets have $r_1, r_2, \cdots, r_t$ vertices is denoted by $K_{r_1, r_2, \cdots, r_t}$. Suppose G and H are two graphs each contains a complete subgraph K_n. Let $G \cup_n H$ denote any graph obtained by overlapping G and H at K_n. In the case that $n = 2$, this is sometimes termed as an *edge-gluing* of G and H.

Suppose G and H are two chromatically unique graphs. While necessary and sufficient conditions for $G \cup_1 H$ to be chromatically unique are already known in the literature (see [3], [13], [15] or [10]), not a great deal is known about the chromatic uniqueness of $G \cup_2 H$. Some necessary conditions were obtained in [3] and [13]. It is asked in [5] (Question 5) whether or not these necessary conditions are also sufficient. No counterexamples are known yet. Some special cases that show that the necessary conditions are also sufficient are given in [2], [4], [6], [8] and [9]. In particular, it is shown in [6] (see also [16]) that $K_{2,s} \cup_2 C_m$ is chromatically unique for all $s \geq 1$ and all $m \geq 3$. The more general situation as whether $K_{r,s} \cup_2 C_m$ is chromatically unique remains unknown.

In this paper, we wish to decide if $K_{n,n,n} \cup_2 C_m$ is chromatically unique. We show that $K_{2,2,2} \cup_2 C_m$ is chromatically unique for all $m \geq 3$ leaving the general case undecided.

Let G be a graph and let A be a subgraph of G. Let $n(A, G)$ denote the number of subgraphs A in G. Let C_n^* denote a chordless cycle on n vertices. Also, let W_n denote the graph (known as the *wheel*) obtained by joining a new vertex to every vertex of C_{n-1}, where $n \geq 4$. Further, let U_n denote the graph obtained from W_n by deleting an edge that joins the new vertex to a vertex of C_{n-1}. The following lemma is a consequence of Theorem 2 of [7].

Lemma 1. *Let G and Y be two graphs such that* $P(G;\lambda) = P(Y;\lambda)$. *Then G and Y have the same number of vertices, edges and triangles. Moreover, in the event that G has no* K_4, *then* $n(C_4^*, G) = n(C_4^*, Y)$ *and*

$$- n(C_5^*, G) + n(K_{2,3}, G) + 2n(U_5, G) + 3n(W_5, G)$$

$$= -n(C_5^*, Y) + n(K_{2,3}, Y) + 2n(U_5, Y) + 3n(W_5, Y).$$

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Then the *cyclomatic number* of G is $|E(G)| - |V(G)| + 1$.

Let $c_k(G)$ denote the coefficient of ω^k in $Q(G;\omega) = P(G;\lambda)$ where $\omega = \lambda - 1$. Notice that if G contains a cut-vertex, then $(\lambda - 1)^2$ divides $P(G;\lambda)$ and hence $c_1(G) = 0$. By Theorem 1 of [14], if G contains no cut-vertices, then $c_1(G) \neq 0$.

Let H be a non-complete graph and let R (respectively T) be any graph obtained by identifying the end-vertices of a path P_m $(m \geq 3)$ with two adjacent (respectively non-adjacent) vertices of H. That is, $R = H \cup_2 C_m$. Let $H + e$ denote the graph obtained by adding a new edge e to two non-adjacent vertices in H, and let $H \cdot e$ be the graph obtained from H by identifying the two end-vertices of this new edge e.

Lemma 2 ([12])**.** $Q(T;\omega) = Q(R;\omega) + (-1)^{m-1} Q(H \cdot e;\omega)$.

Let X_m denote the graph obtained by identifying the end-vertices of a path P_m with two non-adjacent degree-4 vertices of $K_{2,2,2}$ where $m \geq 3$.

Lemma 3. *Let* G *denote the graph* $K_{2,2,2} \cup_2 C_m$ *where* $m \geq 3$. *Then* $|c_1(G)| \neq |c_1(X_m)|$ *and hence* $Q(G;\omega) \neq Q(X_m;\omega)$.

Proof. By applying Lemma 2 to the graph X_m, we have $R = G$, $H = K_{2,2,2}$ and

$$Q(X_m; \omega) = Q(G; \omega) + (-1)^{m-1} Q(K_{1,2,2}; \omega)$$

and the lemma follows by noting that $c_1(K_{1,2,2}) \neq 0$ (because $K_{1,2,2}$ contains no cut-vertices).

Theorem 1. *For any $m \geq 3$, the graph $K_{2,2,2} \cup_2 C_m$ is chromatically unique.*

Proof. Let $G = K_{2,2,2} \cup_2 C_m$. Suppose Y is such that $P(Y; \lambda) = P(G; \lambda)$. Then Y is a 2-connected graph on $m+4$ vertices and $m+11$ edges. Since the graphs $K_{2,2,2} \cup_2 C_3$ and $K_{2,2,2} \cup_2 C_4$ are chromatically unique (see [11]), we may assume that $m \geq 5$.

Note that $n(C_4^*, Y) = 3$ by Lemma 1 because $n(C_4^*, G) = 3$. Furthermore, $n(K_3, Y) = n(K_3, G) = 8$, $n(C_5^*, G) \leq 1$, $n(K_{2,3}, G) = 0$, $n(U_5, G) = 0$ and $n(W_5, G) = 6$. By Lemma 1, it follows that

$$n(K_{2,3}, Y) + 2n(U_5, Y) + 3n(W_5, Y) \geq n(C_5^*, Y) + 17 \geq 17. \tag{1}$$

We assert that Y contains a wheel W_5 as a subgraph. If $n(W_5, Y) \neq 0$, then we are done. Suppose $n(W_5, Y) = 0$. Then we have $n(K_{2,3}, Y) + 2n(U_5, Y) \geq 17$ by Equation (1). If $n(K_{2,3}, Y) \geq 2$, then we have $n(C_4^*, Y) > 3$ which is impossible. Therefore $n(K_{2,3}, Y) \leq 1$ and this implies that $n(U_5, Y) \geq 8$ (by Equation (1)) which further implies that $n(C_4^*, Y) > 3$, a contradiction. Hence Y contains a wheel W_5 as a subgraph. Let W denote this subgraph.

Let J be the graph $Y - W$ and assume that there are e edges joining W to J. Now note that J has $m-1$ vertices and $m+3-e$ edges and so

$$|E(J)| - |V(J)| = 4 - e. \tag{2}$$

Let $J_1, \cdots, J_k$ be the connected components of J, $k \geq 1$. Suppose there are e_i edges joining W and J_i, $i = 1, \cdots, k$ so that $e = \sum_{i=1}^k e_i$. Let c_i denote the cyclomatic number of J_i, $i = 1, \cdots, k$. Let $c = \sum_{i=1}^k c_i$. Using Equation (2) and from the definition of cyclomatic number, we have

$$c = \sum_{i=1}^{k} c_i = 4 - e + k. \tag{3}$$

Some observations are in order. Since G contains no cut-vertices, the same is true for Y. Consequently, we see that

(O1) $e_i \geq 2$ for every $i = 1, \cdots, k$.

(O2) For every $i = 1, \cdots, k$, there exists no vertex in J_i which is adjacent to e_i vertices of W unless J_i is an isolated vertex. Likewise, there exists no vertex in W that is adjacent to e_i vertices in J_i.

(O3) If $c_i = 0$, then J_i is either an isolated vertex or else a tree with at most e_i vertices of degree 1.

We also note the following.

(O4) The cyclomatic number of G (and so of Y) is 8 and that the cyclomatic number of W is 4.

(O5) W contains precisely four K_3 and one C_4^*.

(O6) Since the chromatic number of G (and so of Y) is 3, it follows that Y contains no K_4 as subgraph.

For each $i = 1, \cdots, k$, let α_i denote the number of triangles in the subgraph induced by J_i and those edges (together with their end-vertices) that join J_i to W. Let $\alpha = \sum_{i=1}^{k} \alpha_i$ and let β denote the number of C_4^* in Y not including the one in W. Since $n(K_3, Y) = 8$ and $n(C_4^*, Y) = 3$, we have $\alpha = 4$ and $\beta = 2$ by (O5).

Using (O1) and Equation (3), we see that $c \leq 4 - k$. Since $c \geq 0$, it follows that $k \leq 4$. In the rest of the proof, we shall consider the various cases of k. We shall show that either $Y \cong K_{2,2,2} \cup_2 C_m$ or else $\alpha \neq 4$ or $\beta \neq 2$ or $|V(Y)| \leq 8$.

Suppose $k = 4$. Then it follows from the fact that $c \leq 4 - k$ and Equation (3) that $c = 0$ and $e = 8$ so that $e_i = 2$ and $c_i = 0$ for each $i = 1, \cdots, 4$. By (O3), J_i is either an isolated vertex or else a path P_n with $n \geq 2$. If some J_i is not an isolated vertex, then $\alpha < 4$. If all the J_i's are isolated vertex, then either $\alpha < 4$ or else $\beta < 2$. Either case is a contradiction.

Suppose $k = 3$. Then it follows from Equation (3) that $c = 7 - e$. By (O1), we see that $0 \leq c \leq 1$ and that $6 \leq e \leq 7$.

If $c = 0$, then $c_i = 0$ for all $i = 1, 2, 3$ and $e = 7$. We may assume without loss of generality that $e_1 = 3$, $e_2 = 2 = e_3$. In this case, each J_i admits no triangles. In order to achieve $\alpha = 4$ and $\beta = 2$, each of J_2 and J_3 must be an isolated vertex and is adjacent to two adjacent vertices of W so that $\alpha_2 = 1 = \alpha_3$. Further, $\alpha_1 = 2$ and $\beta = 2$. For this to be possible, J_1 must be an isolated vertex adjacent to three vertices (which form a path of length 3) in W. But then $|V(Y)| = 8$, a contradiction since $m \geq 5$.

If $c = 1$, we may take $c_1 = 1$ and $c_2 = 0 = c_3$ without loss of generality. From Equation (3), $e = 6$ and we have $e_i = 2$ for all $i = 1, 2, 3$. Since J_1 is not an isolated vertex, the two edges joining J_1 and W yields no triangle by ($O2$) so that $\alpha_1 \leq 1$. Since $\alpha_2 \leq 1$ and $\alpha_3 \leq 1$, we have $\alpha \leq 3$, a contradiction.

Suppose $k = 2$. Then $c \leq 2$. Now since $|V(Y)| \geq 9$, we have

$$|V(J)| = |V(J_1)| + |V(J_2)| \geq 4. \tag{4}$$

Suppose $c = 0$. Then $c_1 = 0 = c_2$ and each J_i admits no triangles. Since $e = 6$, there are two possible cases. Either (i) $e_1 = 3 = e_2$ or else (ii) $e_1 = 4$ and $e_2 = 2$.

(i) If $|V(J_i)| \geq 2$, $i = 1, 2$, then we have $\alpha < 4$ or $\beta < 2$. Hence we may assume that $|V(J_1)| = 1$ and $|V(J_2)| \geq 3$. But then, we have $\alpha < 4$ because $\alpha_1 \leq 2$ and $\alpha_2 \leq 1$.

(ii) If $|V(J_1)| \geq 2$, then we do not have $\alpha = 4$ and $\beta = 2$ unless $|V(J_1)| = 2$ and $|V(J_2)| = 1$ (but this violates Inequality (4)). Therefore J_1 is an isolated vertex which must be adjacent to the four vertices of degree 3 of W (by taking note of ($O6$)). Further, J_2 is a path P_{m-2}. If the end-vertices of J_2 are adjacent to two non-adjacent vertices of W, then $Y \cong X_m$, in which case, $P(Y;\lambda) \neq P(G;\lambda)$ by Lemma 3. Hence the two end-vertices of J_2 are adjacent to two adjacent vertices of W, in which case, $Y \cong K_{2,2,2} \cup_2 C_m$.

Suppose $c = 1$. Then we may assume that $c_1 = 1$ and $c_2 = 0$ without loss of generality. Note that J_1 admits at most one triangle. Since $e = 5$, there are two possible cases. Either (iii) $e_1 = 3$ and $e_2 = 2$ or else (iv) $e_1 = 2$ and $e_2 = 3$.

(iii) In this subcase, $\alpha_1 \leq 3$ and $\alpha_2 \leq 1$. If equality holds for both α_1 and α_2, then, (by taking note of ($O2$)), we have $\beta < 2$.

(iv) In this subcase, $\alpha \leq 3$ because $\alpha_1 \leq 1$ and $\alpha_2 \leq 2$.

Suppose $c = 2$. Then $e = 4$. Clearly $e_1 = 2 = e_2$. There are two possible cases. Either (v) $c_1 = 2$ and $c_2 = 0$ or else (vi) $c_1 = 1 = c_2$.

(v) In this subcase, $\alpha \leq 3$ because $\alpha_1 \leq 2$ (as J_1 admits at most two triangles) and $\alpha_2 \leq 1$.

(vi) In this subcase, $\alpha \leq 2$ because $\alpha_1 \leq 1$ and $\alpha_2 \leq 1$ (as each J_i admits at most one triangle).

Suppose $k = 1$. Then $c \leq 3$.

If $c = 0$, then $e = 5$. In this subcase, $J_1 = J$ is a tree. If J_1 contains three or more end-vertices, then $\alpha = \alpha_1 \leq 2$. Therefore J_1 is a path P_{m-1}. Since $\alpha = 4$, one of the end-vertex of J_1 is adjacent to the four vertices of degree 3 of W (by taking note of (O6)). If the other end-vertex of J_1 is adjacent to the degree-4 vertex of W, then $Y \cong X_m$, in which case, $P(Y;\lambda) \neq P(G;\lambda)$ by Lemma 3. Hence the other end-vertex of J_1 is adjacent to a degree-3 vertex of W, in which case, $Y \cong K_{2,2,2} \cup_2 C_m$.

If $c = 1$, then J_1 admits at most one triangle and $e = 4$. Since $|V(J_1)| \geq 4$, we have $\alpha \leq 3$ or $\beta < 2$.

If $c = 2$, then J_1 admits at most two triangles and $e = 3$. Again we have $\alpha \leq 3$ or $\beta < 2$.

If $c = 3$, then J_1 admits at most three triangles (because Y contains no K_4 as subgraph by (O6)). Since $e = 2$, by (O2), we have $\alpha \leq 3$.
This completes the proof.

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