BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **26** (2003) 209–219

On a Class of Residually Finite Groups

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Abstract. Let n, k be positive integers and $t_0, t_1, \dots, t_k$ be non-zero integers. We denote by $\overline{W}_k(n)$ the class of groups G in which, for every subset X of G of cardinality $n+1$, there exist a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$, , and a function $f : \{0, 1, 2, \dots, k\} \to X_0$, with $f(0) \neq f(1)$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$ where $x_i := f(i)$, $i = 0, 1, \dots, k$. The class $\overline{W}_k^*(n)$ is defined exactly as $\overline{W}_k(n)$, with additional conditions " $x_j \in H$ whenever $x_j^{t_j} \in H$, where $\langle x_j^{t_j} \rangle \neq H \leq G$ ".

Let G be a finitely generated residually finite group. Here we prove that

(1) If $G \in \overline{W}_k(n)$, then G has a normal nilpotent subgroup N with finite index such that the nilpotency class of N / N_t is bounded by a function of k, where N_t is the torsion subgroup of N.

(2) If $G \in \overline{W}_k^*(n)$ be d generated, then G has a normal nilpotent subgroup N whose index and the nilpotency class are bounded by a function of $k, n, t_0, t_1, \dots, t_k$.

1. Introduction and results

In response to a question of Paul Erdös, B.H. Neumann proved [16] that a group is center-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. Other problems of this type have been the object of several articles, for example [1] – [13], [16], [20] – [23].

Our notation and terminology are standard and can be found in [17]. In particular for a group G and elements $x, y, x_1, x_2, \cdots, x_k \in G$ we write

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 = x_1^{-1} x_1^{x_2}, \quad [x_1, \dots, x_k] = [[x_1, \dots, x_{k-1}], x_k],$$
$$[x,_0 y] = x, \ [x,_k y] = [[x,_{k-1} y], y].$$

A group is said to be a k-Engel group (respectively, Engel group) if for all $x, y \in G$, $[x,_k y] = 1$ (respectively, there exists a positive integer t depending on x and y such that $[x,_t y] = 1$). The class of k-Engel (respectively, Engel) groups will be denoted by ε_k (respectively, ε).

Let k be a positive integer, n be a positive integer or infinity (denoted by ∞). We denote by $\varepsilon_k(n)$ (respectively, $\varepsilon(n)$) the class of all groups G such that for every subset X of cardinality $n+1$, there exist distinct elements $x, y \in X$ such that $[x,_k y] = 1$ (respectively, $[x,_t y] = 1$ for some positive integer t depending on x, y). Longobardi and Maj [12] (see also [8]) proved that a finitely generated soluble group G has the property $\varepsilon(\infty)$ if and only if G is finite-by-nilpotent. Abdollahi [2], showed that finite $\varepsilon(2)$-groups (respectively, $\varepsilon(15)$-groups) are nilpotent (respectively, soluble) and that a finitely generated residually finite $\varepsilon_k(n)$-group, n, k are positive integers, is finite-by-nilpotent. In [21] the class of all groups G in which for every subset X of cardinality $n+1$, there exist a positive integer k and distinct elements $x, y \in X$ and non-zero integers $t_0, t_1, \dots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}$, $x_0 \neq x_1$, is denoted by $\Omega(n)$. The author in [21] proved that a finitely generated soluble group G has the property $\Omega(\infty)$ if and only if G is nilpotent-by-finite.

Now, in order to generalize the classes of groups mentioned above, we define a new class of groups as follows. Let n be a positive integer or infinity. We denote by $W(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exists a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$, such that the following condition holds.

There exist a positive integer k and a function $f : \{0,1,2,\dots,k\} \to X_0$ and non-zero integers $t_0, t_1, \dots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$ where $x_i := f(i)$, $i = 0,1,\dots,k$ and $x_0 \neq x_1$.

If k is fixed for every subset X then we obtain the class $W_k(n)$. Clearly the classes $W(n)$ and $W_k(n)$ are subgroup and quotient closed. Since all torsion groups are belong to $W(n)$ we define another class of groups: the class $W^*(n)$ is defined exactly as $W(n)$ with additional conditions "$x_j \in H$ whenever $x_j^{t_j} \in H$, for some subgroup $H \neq \langle x_j^{t_j} \rangle$ of G". It is clear that the class $W^*(n)$ is subgroup and quotient closed. In fact we impose the above condition to ensure that the class $W^*(n)$ be quotient closed. To see this let K be a normal subgroup of a $W^*(n)$-group G, and let $X = \{g_0K, g_1K, \dots, g_nK\}$ be a subset of G/K of size $n+1$. Since $G \in W^*(n)$, there exists a positive integer k, and non-zero integers $t_0, t_1, \dots, t_k$, such that

$[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{g_0, g_1, \dots, g_k\}$, $x_0 \neq x_1$, and if $x_j^{t_j} \in H$, where $\langle x_j^{t_j} \rangle \neq H \leq G$, then $x_j \in H$. Now $[(x_0K)^{t_0}, (x_1K)^{t_1}, \dots, (x_kK)^{t_k}] = K$, $x_0K \neq x_1K$ and if $(x_jK)^{t_j} \in H/K \leq G/K$, where $H/K \neq \langle (x_jK)^{t_j} \rangle$, then we have $x_j^{t_j} \in H$ and $\langle x_j^{t_j} \rangle \neq H \leq G$. So $x_j \in H$ and $x_jK \in H/K$. Therefore $G/K \in W^*(n)$.

Similarly we can define $W_k^*(n)$. The classes $W(n)$, $W^*(n)$ are considered in [22]. In that paper the author proved that a finite group with the property $W^*(2)$ (respectively $W^*(4)$) is nilpotent (respectively soluble), and that a finitely generated soluble group G has the property $W^*(\infty)$ if and only if G is finite-by-nilpotent.

Now we define another classes of groups, which we want to consider in this paper, namely the classes $\overline{W_k}(n)$ and $\overline{W_k^*}(n)$: if k and $t_0, t_1, \dots, t_k$, in the definition of $W(n)$ (respectively $W^*(n)$) are the same for every subset X_0, one obtains the class $\overline{W_k}(n)$ (respectively $\overline{W_k^*}(n)$). Note that we just fix k and $t_0, t_1, \dots, t_k$, so that the bar notation causes no confusion here.

If the subset X_0 in the definition of $\overline{W_k}(n)$ (respectively, $\overline{W_k^*}(n)$) has always 2 elements, say $X_0 = \{x, y\}$, and the function f is always of the form $f(0) = x$ and $f(i) = y$ and $t_0 = t_1 = \cdots = t_k = 1$ one obtains the class $\varepsilon_k(n)$. Note that

$$\varepsilon_k(n) \subseteq \overline{W_k^*}(n) \subseteq \overline{W_k}(n) \subseteq W_k(n) \subseteq W_k(n+1).$$

Throughout the paper we assume that n, k are fixed positive integers and $t_0, t_1, \dots, t_k$ are non-zero fixed positive integers. In this paper we consider residually finite groups. Our first result is about $\overline{W_k}(n)$-groups and sharpens [21, Theorem 3]:

Theorem A. *Let G be a finitely generated residually finite $\overline{W_k}(n)$-group. Then G has a normal nilpotent subgroup N with finite index such that the nilpotency class of N/N_t is bounded by a function of k, where N_t is the torsion subgroup of N.*

If we consider the stronger condition $\overline{W_k^*}(n)$ we are able to prove the stronger result:

Theorem B. *Let G be a d generated residually finite $\overline{W_k^*}(n)$-group. Then G has a normal nilpotent subgroup N whose index and the nilpotency class are bounded by a function of $k, n, t_0, t_1, \dots, t_k$.*

2. Proof of Theorem A

To consider finitely generated residually finite group in $\overline{W_k}(n)$ we can use a result of Wilson [24], which states that if G is finitely generated residually finite group and N is positive integer such that G has no section isomorphic to the twisted wreath product $A \operatorname{twr}_C B$, with B is finite and cyclic, A an elementary Abelian group acted on faithfully and irreducibly by C, and $|B : C| > N$, then G is virtually a soluble minimax group. For the definition of the twisted wreath products we refer readers to Neumann [15]. Now as in the proof of the Lemma 5 in [21], we can see that

Lemma 2.1. *Let A be a non-trivial Abelian group, $B = \langle b \rangle$ a finite cyclic group of order n, C a subgroup of B with $|B : C| = N$, and suppose that C acts on A. Let $W = A \operatorname{twr}_C B$ be the twisted wreath product of A by B with respect to the action of C on A. If $G \in \overline{W_k}(n)$, then $N \leq n + t_0 + t_1 + \cdots + t_k$.*

Lemma 2.1 and the result of Wilson reduces the residually finite case to the soluble case. For finitely generated soluble groups we may argue exactly as in the proof of [21, Theorem 3], and prove the following.

Proposition 2.2. *Let $G \in \overline{W_k}(n)$, be a finitely generated soluble group. Then G is nilpotent-by-finite.*

Now we prove that nilpotency class of torsion free nilpotent $\overline{W_k}^*(n)$-groups is k-bounded. The following lemma is proved in [22], but we include it for completeness.

Lemma 2.3. *Let $G \in \overline{W_k}(n)$, be torsion free nilpotent. Then the nilpotency class of G is bounded by a function of k.*

Proof. Let G be nilpotent of class c. Then $\Gamma_{[c/2]}(G)$ is Abelian, where $[c/2]$ equals $(c+2)/2$ if c is even and $(c+1)/2$ if c is odd ($\Gamma_{s+1}(G)$ is the sth term of lower central series of G). Let $A = \{x \in G \mid x^m \in \Gamma_{[c/2]}(G)$, for some non-zero integer $m\}$ denote the isolator of $\Gamma_{[c/2]}(G)$. Then A is Abelian, since G is torsion free. Now let $a \in A$ and $g \in G$. Considering the elements $ag, ag^2, ag^3, \ldots$ we find positive integers $i_0, i_1, \ldots, i_k$, $i_0 \neq i_1$, such that

$$1 = [(a^{i_0} g)^{t_0}, (a^{i_1} g)^{t_1}, \ldots, (a^{i_k} g)^{t_k}] = [(a^{i_0} g)^{t_0}, (a^{i_1} g)^{t_1}, g^{t_2}, \ldots, g^{t_k}].$$

Now, since G is metabelian, $[a,u,v]=[a,v,u]$, for all $a\in G'$ and $u,v\in G$. It follows that in the above equation we may replace t_j by $|t_j|$, for $j\geq 2$. Now suppose that $t_0<0$. Then since A is Abelian normal, we have

$$[(a^{i_0}g)^{-t_0},(a^{i_1}g)^{t_1}]=[(a^{i_0}g)^{t_0},(a^{i_1}g)^{t_1}]^{-(a^{i_0}g)^{t_0}}$$
$$=[(a^{i_1}g)^{t_1},(a^{i_0}g)^{t_0}]^{g^{-t_0}a^{i_0(g^{-t_0}+g^{-t_0-1}+\cdots+g)}}$$
$$=[(a^{i_1}g)^{t_1},(a^{i_0}g)^{t_0}]^{g^{-t_0}}.$$

Therefore, since the identity $[x_1,x_2,x_3,\ldots,x_k]^{-1}=[x_2,x_1,x_3,\ldots,x_k]$ holds in a metabelian group, we have

$$[(a^{i_0}g)^{-t_0},(a^{i_1}g)^{t_1},g^{t_2},\ldots,g^{t_k}]=[(a^{i_1}g)^{t_1},(a^{i_0}g)^{t_0},g^{t_2},\ldots,g^{t_k}]^{g^{-t_0}}$$
$$=[(a^{i_0}g)^{t_0},(a^{i_1}g)^{t_1},g^{t_2},\ldots,g^{t_k}]^{-g^{t_0}}$$
$$=1^{-g^{-t_0}}=1.$$

Thus we may replace t_0 by $|t_0|$. Similarly we may replace t_1 by $|t_1|$. Hence we may assume that $t_1>0$, for all $i=0,1,\ldots,k$. We treat A as a $\boldsymbol{Z}\langle g\rangle$-module and show that $A(g-1)^N=0$, where N is a function of k. If $g\in A$, then $A(g-1)=0$ and we are done. So suppose that $g\notin A$. Now, since

$$\left[g^{t_0}a^{i_0(g^{t_0}+g^{t_0-1}+\cdots+g)},\ g^{t_0}a^{i_1(g^{t_1}+g^{t_1-1}+\cdots+g)},\ g^{t_2},\ldots,\ g^{t_k}\right]=1,$$

we have $af_1(g)=0$, where

$$f_1(x)=\left(i_1(x^{t_1}+x^{t_1-1}+\cdots+x)(x^{t_0}-1)+i_0(x^{t_0}+x^{t_0-1}+\cdots+x)(x^{t_1}-1)\right)$$
$$(1-x^{t_2})\cdots(1-x^{t_k}).$$

Put $f(x):=(x-1)f_1(x)=(i_1+i_0)x(x^{t_0}-1)(x^{t_1}-1)(1-x^{t_2})\cdots(1-x^{t_k})$. Then $af(g)=0$ and $f(x)=\sum_{i=1}^{t}c_ix^{m_i}$, where $c_i\neq 0$, $t\leq N$, and N is a function of k. Let $A_1=A\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. We consider g as an operator on A_1, and obtain that $af(g)=0$. Since $\langle A_1,g\rangle$ is also nilpotent of class at most c, $(g-1)^c$ annihilates a, as $f(g)$ does. Now if $(x-1)^e$ divides $f(x)$, then $f(1)=f'(1)=\cdots=f^{(e-1)}(1)=0$. Thus

$$\begin{cases} c_1 + \cdots + c_t = 0 \\ m_1 c_1 + \cdots + m_t ct = 0 \\ \vdots \qquad \vdots \qquad \vdots \\ m_1^{e-1} c_1 + \cdots + m_t^{e-1} c_t = 0. \end{cases}$$

Note that $0 = f'(1) = m_1(m_1 - 1)c_1 + \cdots + m_t(m_t - 1)c_t$ implies that $m_1^2 c_1 + \cdots + m_t^2 c_t = 0$, since $m_1 c_1 + \cdots + m_t c_t = 0$, and so on. If $e \geq N$, then $e \geq t$ and since the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ m_1 & m_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{t-1} & m_2^{t-1} & \cdots & m_t^{t-1} \end{bmatrix}$$

is invertible, the only solution of the system is $c_i = 0$, for all $i = 0, 1, \ldots, t$. This means that $f(x) = 0$, a contradiction. Therefore $e < N$. Thus $a(g-1)^N = 0$, for all $a \in A$ and $g \in G$. In multiplicative notation of the group G, we have

$$[A, \underbrace{g, \ldots, g}_{N}] = 1.$$

Since G is torsion free, a result of Zel'manov (see [25 p.166]) implies that, A lies in $Z_{\mu(N)}(G)$, the $\mu(N)$ th center of G, where $\mu(N)$ is a function of N and independent of the number of generators of G. Thus the nilpotency class of G is at most $[c/2] + \mu(N)$ and hence $c \leq 2\mu(N)$.

Proof of the Theorem A. By Lemma 2.1 G has no section isomorphic to $W = A \operatorname{twr}_C B$, where A elementary Abelian, B finite cyclic, C a subgroup of B in which acts faithfully irreducibly on A, such that $|B : C| > n + t_0 + t_1 + \cdots + t_k$. Thus by a result of Wilson [24], G is virtually a soluble minimax group. Hence there exist a normal subgroup H of G with finite index, such that H in a soluble minimax group. By Proposition 2.2 H nilpotent-by-finite and so is G. Thus there exists a normal nilpotent subgroup N of G such that G/N is finite. Now the nilpotency class of N/N_t is bounded by a function of k, by Lemma 2.3, so the result follows.

3. Proof of Theorem B

To prove Theorem B we follow the arguments given in [19] and obtain the corresponding results.

Lemma 3.1. *The variety* $V_{n,k,t_0,t_1,\ldots,t_k}$ *generated by all* $\overline{W_k}(n)$*-groups is not the variety of all groups.*

Proof. Let $x_0, x_1, \ldots, x_n$ be letters. Let $w_1, w_2, \ldots, w_r$ be the list of all commutators of weight $k+1$ of the form $[y_0^{t_0}, y_1^{t_1}, \ldots, y_k^{t_k}] = 1$ where $y_i \in \{x_0, x_1, \ldots, x_n\}$, and $y_0 \neq y_1$. Let $w = [w_1, w_2, \ldots, w_r]$, then w is a non-trivial word. It is clear that if $G \in \overline{W_k}(n)$ then G satisfies the law $w = 1$. So the Lemma is proved.

Now as in the proof of [19, Lemma 2.2] we have

Lemma 3.2. *For all positive integers* n, k *and non-zero integers* $t_0, t_1, \ldots, t_k$ *there exists a positive integer* $s(n, k, n, t_0, t_1, \ldots, t_k)$ *such that if* S *is a finite simple* $\overline{W_k}^*(n)$*-group, then* $|S| \leq s(n, k, t_0, t_1, \ldots, t_k)$.

Therefore as in the proof of [19, Proposition 2.4], where we use Lemma 2.1 and Lemma 3.2 instead of Lemma 2.3 and Lemma 2.2 of [19], respectively, we have

Proposition 3.3. *For all positive integers* n, k *and non zero integers* $t_0, t_1, \ldots, t_k$ *there exists a positive integer* $h(n, k, n, t_0, t_1, \ldots, t_k)$ *such that if* G *is any finite* $\overline{W_k}^*(n)$*-group, then* G *has a soluble characteristic subgroup* H *such that* $\exp(G/H)$ *divides* $h(n, k, n, t_0, t_1, \ldots, t_k)$.

According to the Proposition 3.3 we must consider finite soluble $\overline{W_k}^*(n)$-group. But firstly we prove the following Lemma, which is similar to the [18, Lemma 3.3] and stated in the proof of [19, Lemma 3.1].

Lemma 3.4. *Let* $f \in \boldsymbol{F}_p[x]$ *be polynomial of degree* d, *and let* $n_0, n_1, \ldots, n_d$ *be* $d+1$ *distinct positive integers. Let* I *be the ideal generated by* $f(x^{n_0}), f(x^{n_1}), \ldots, f(x^{n_d})$ *in* $\boldsymbol{F}_p[x]$. *Then* $(x^m - 1)^r \in I$ *for some positive integer* r, m.

Proof. Let M be any maximal ideal of $\boldsymbol{F}_p[x]$, and consider the field $F = \boldsymbol{F}_p[x]/M$. By the definition of I and M we have $f(\overline{x}^{n_i}) = \overline{0}$, $i = 0,1,\ldots,d$, where $\overline{u}$ denotes the image of $u \in \boldsymbol{F}_p[x]$ in F. But f cannot have more than $d = \deg(f)$ distinct roots in F. Hence $\overline{x}^{n_i} = \overline{x}^{n_j}$ for some $n_i < n_j$, which implies $\overline{x}^c = \overline{1}$, where $c = n_j - n_i$. Thus $x^c - 1 \in M$. Now $\sqrt{I}$, the radical of I, is the intersection of finitely many maximal ideal, $M_1, M_2, \ldots, M_s$ say, and we have obtained positive integers c_i, $i = 1,2,\ldots,s$, such that $x^{c_i} - 1 \in M_i$. If $m = \prod_{i=1}^{s} c_i$, we get $x^m - 1 \in \bigcap_{i=1}^{s} M_i = \sqrt{I}$. Therefore there exists positive integer r such that $(x^m - 1)^r \in I$, as required.

Observe that if $a, b, a_0, a_1, \ldots, a_k \in A$, where A is Abelian normal subgroup of a group G and $u, v, x \in G$, and i_j, t_j, t, s are positive integers, then

$$[(a_0^{x^{i_0}} x)^{t_0}, (a_1^{x^{i_1}} x)^{t_1}, \ldots, (a_k^{x^{i_k}} x)^{t_k}] = [(a_0^{x^{i_0}} x)^{t_0}, (a_1^{x^{i_1}} x)^{t_1}, x^{t_2}, \ldots, x^{t_k}]$$
$$= [x^{t_0} a_0^{x^{i_0}(x^{t_0}+\cdots+x)}, x^{t_1} a_1^{x^{i_1}(x^{t_1}+\cdots+x)}, x^{t_2}, \ldots, x^{t_k}]$$

and

$$[x^t a^u, x^s b^v] = a^{u(1-x^t)+v(x^s-1)}.$$

We use these facts in the following Lemma.

Lemma 3.5. *Let $G = \langle g \rangle \propto A$, be the splitting extension of an Abelian normal subgroup A by a cyclic subgroup $\langle g \rangle$. If $G \in \overline{W_k}(n)$ then, $[A,_r g^m] = 1$, for some positive integers r, m depending only on $n, t_0, t_1, \ldots, t_k$.*

Proof. Fix $a \in A$ and first suppose that $t_0, t_1, \ldots, t_k$ are positive. Consider the elements $a^g g, a^{g^2} g, \ldots, a^{g^{n+1}} g$. Then, since $G \in \overline{W_k}^*(n)$, there exist $1 \le i_0, i_1, \ldots, i_k \le n+1$, such that

$$1 = [(a^{g^{i_0}} g)^{t_0}, (a^{g^{i_1}} g)^{t_1}, \ldots, (a^{g^{i_k}} x)^{t_k}] = [g^{t_0} a^{g^{i_0}(g^{t_0}+\cdots+g)}, g^{t_1} a^{g^{i_1}(g^{t_1}+\cdots+g)}, g^{t_2}, \ldots, g^{t_k}].$$

Now, since G is metabelian, $[a,u,v] = [a,v,u]$, for all $a \in G'$ and $u, v \in G$. It follows that in the above equation we may replace t_j by $|t_j|$, for $j \ge 2$. Let $b = a^{g^{i_0}(g^{t_0}+\cdots+g)}$ and $c = a^{g^{i_1}(g^{t_1}+\cdots+g)}$, and $t_0, t_1 > 0$. Then $(a^{g^{i_0}} g)^{-t_0} = ((a^{g^{i_0}} g)^{t_0})^{-1} = g^{t_0} b^{-1}$ and so

$$\begin{aligned}[(a^{i_0}g)^{-t_0},(a^{i_1}g)^{t_1}] &= [(g^{t_0}b)^{-1},g^{t_1}c] \\ &= [g^{t_0}b,g^{t_1}c]^{-(g^{t_0}b)^{-1}} \\ &= [g^{t_1}c,g^{t_0}b]^{g^{-t_0}}.\end{aligned}$$

Therefore $1=[(g^{t_0}b)^{-1},g^{t_1}c,g^{t_2},\dots,g^{t_k}]^{g^{-t_0}}$, and since in a metabelian group we have

$$[g_1,g_2,g_3,\dots,g_k]^{-1}=[g_2,g_1,g_3,\dots,g_k],$$

we obtain that $[g^{t_0}b,g^{t_1}c,g^{t_2},\dots,g^{t_k}]=1$. Thus we have shown that we may replace t_0 by $|t_0|$. Similarly we may replace t_1 by $|t_1|$. Therefore if we put

$$f(x)=\left(x^{i_0}(x^{|t_0|}+x^{|t_0|-1}+\cdots+x)-x^{i_1}(x^{|t_1|}+x^{|t_1|-1}+\cdots+x)\right)(1-x^{|t_2|})\cdots(1-x^{|t_k|})$$

we have, in additive notation, that $a\cdot f(g)=0$. Note that since $1\le i_0,i_1\le n+1$,

$$\deg(f)\le d=(n+1)\max\{|t_0|,|t_1|\}+\sum_{i=2}^{k}|t_i|.$$

Denote the polynomial f just constructed by $f_1\in \boldsymbol{F}_p[x]$. By applying the same argument for g^m instead of g we obtain a similar polynomial $f_m(x)\in \boldsymbol{F}_p[x]$ with the property that $a\cdot f_m(g^m)=0$. Observe that there are not more than $(n+1)^2$ possibilities for the polynomial f_m, since $t_0,t_1,\dots,t_k$ are fixed and i_0,i_1 may vary in $\{1,2,\dots,n+1\}$. Letting m range between 1 and $N:=1+(n+1)^2$, we conclude that some polynomial f is obtained, more than d times in this way, namely $f=f_{m_1}=f_{m_2}=\cdots=f_{m_{d+1}}$, where $m_i\le N$, are distinct positive integers, and $a\cdot f(x^{m_i})=0$, $i=1,2,\dots,d+1$.

Let $I\subseteq \boldsymbol{F}_p[x]$ be the ideal generated by above polynomial $f(x^{m_i})$. Applying Lemma 3.4 we conclude that some polynomial of the form $(x^m-1)^r$ lies in I. Moreover since given $n,t_0,t_1,\dots,t_k$, there are boundedly many possibilities for f and the integers $m_1,m_2,\dots,m_{d+1}$, the parameters r,m are bounded in terms of $n,t_0,t_1,\dots,t_k$, say $r\le r(n,t_0,t_1,\dots,t_k)$, $m\le m(n,t_0,t_1,\dots,t_k)$. This implies that $(x^m-1)^r\in I$, where $r:=r(n,t_0,t_1,\dots,t_k)!$, $m:=m(n,t_0,t_1,\dots,t_k)!$.

Since I is generated by polynomials which act trivially on a, $(g^m - 1)^r$ acts trivially on a. However the choice of r, m is independent of a. Hence in $End(A)$ we have $(g^m - 1)^r = 0$, and in multiplicative notation, we have $[a,_r g^m] = 1$, for all $a \in A$.

Corollary 3.6. *Let* $G = \langle A, g \rangle$, *where* A *is an elementary Abelian normal* p*-subgroup of* G, p *a prime. If* $G \in \overline{W}_k^*(n)$, *then* $[A,_r g^m] = 1$, *for some positive integers* r, m *depending only on* $n, t_0, t_1, \ldots, t_k$.

Proof. Obviously $G/(A \cap < g >)$ is the splitting extension of $A/(A \cap < g >)$ by $< g > /(A \cap < g >)$. Thus, by Lemma 3.5, there exist positive integers s, m depending only on $n, k, t_0, t_1, \ldots, t_k$, such that $[A,_s g^m] \in A \cap < g >$. So $1 = [A,_s g^m, g^m] = [A,_{s+1} g^m]$.

If we use Corollary 3.6 and follow the proof of the [19, Corollary 3.3] we obtain that

Corollary 3.7. *Let* G *be finite soluble* $G \in \overline{W}_k^*(n)$ *-group, then there exists a positive integer* $e = e(n, k, t_0, t_1, \ldots, t_k)$ *such that* G^e *is nilpotent.*

Using the above results and arguing exactly as the [19, sections 4, 5] we obtain that

Theorem 3.8. *There exist functions* f, g *such that every finite group* $G \in \overline{W}_k^*(n)$ *possesses a nilpotent normal subgroup* N *satisfying*

(1) $\exp(G/N)$ *divides* $f(n, k, t_0, t_1, \ldots, t_k)$, *and*
(2) every d-generator subgroup of N *has class at most* $g(n, k, t_0, t_1, \ldots, t_k)$.

Now the proof of Theorem B follows from argument given in [19, page 52].

Acknowledgement. The author was supported in part by Isfahan University of Technology.

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2000 AMS Mathematics Subject Classification: 20F99