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A Note on Some Results of Schwick

YAN XU AND MINGLIANG FANG
Department of Mathematics, Nanjing Normal University, Nanjing 210097, P.R. China
e-mail: xuyan@njnu.edu.cn and e-mail: mlfang@pine.njnu.edu.cn

Abstract. In this paper, we obtain some normality criteria of families of meromorphic functions, which improve and generalize the related results of Schwick [8, 10]. Some examples are given to show the sharpness of our results.

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1. Introduction

Let D be a domain in $\boldsymbol{C}$ and $a \in \boldsymbol{C}$ and let S be a set of complex numbers. For f meromorphic on D, set

$$\overline{E}(a,\ f) = \{z : z \in D,\ f(z) = a\}.$$

$$\overline{E}(S,\ f) = \{z : z \in D,\ f(z) \in S\}.$$

Two meromorphic functions f and g are said to share the value a in D if $\overline{E}(a,\ f) = \overline{E}(a,\ g)$. Similarly, f and g are said to share the set S in D if $\overline{E}(S,\ f) = \overline{E}(S,\ g)$.

Let F be a family of meromorphic functions defined in D. F is said to be normal in D, in the sense of Montel (see Schiff [7]), if, for any sequence $f_n \in F$, there exists a subsequence f_{n_j}, such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

Schwick [8] seems to have been the first to draw a connection between normality criteria and shared values. He proved

Theorem A. *Let F be a family of meromorphic functions defined in D, and let a, b, c be three distinct complex numbers. If f and f' share a, b, c in D for every $f \in F$, then F is normal in D.*

This result has undergone various extensions [3,11,12], culminating in the following results due to Pang and Zalcman [5], and Chen and Fang [1], respectively.

Theorem B. ([5]) *Let F be the family of meromorphic functions in a domain D, and let a, b be two distinct complex numbers. If f and f' share a and b in D for each $f \in F$, then F is normal in D.*

Theorem C. ([1]) *Let F be the family of meromorphic functions in a domain D, k a positive integer, and let a, b and c be complex numbers such that $a \neq b$. If, for each $f \in F$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geq k+1$, then F is normal in D.*

Remark 1. There is an example (see [1]) to show that the assumption on the zeros of $f(z) - c$ is required for Theorem C to hold.

In this paper, we obtain the following results.

Theorem 1. *Let F be the family of meromorphic functions in a domain D, let a, b and c be three distinct complex numbers, and let $S_1 = \{a, b\}$ and $S_2 = \{c\}$, if, for each $f \in F$, f and $f^{(k)}$ share the set S_1 and S_2 in D, then F is normal in D.*

Theorem 2. *Let F be the family of meromorphic functions in a domain D, let $a_i (i = 1, 2, 3)$ be three distinct complex numbers, if, for each $f \in F$, $f = a_i \Rightarrow f' = a_i (i = 1, 2, 3)$ in D, then F is normal in D.*

The second part of this paper is concerning on the result of Schwick [10]. In 1983, Yang [13] (see also [9]) proved

Theorem D. *Let $\psi \not\equiv 0$ be a analytic function in a domain D and $k \in \mathbf{N}$. Let F be the family of meromorphic functions in D such that f and $f^{(k)} - \psi$ have no zeros for each $f \in F$, then F is normal in D.*

In 1997, Schiwick extended ψ to meromorphic case in Theorem *D*, as follows.

Theorem E. ([10]) *Let $\psi \not\equiv 0$ be a meromorphic function in a domain D and $k \in \mathbf{N}$. Let F be the family of meromorphic functions in D such that f and $f^{(k)} - \psi$ have no zeros and f and ψ have no common poles for each $f \in F$, then F is normal in D.*

It is natural to ask: whether or not the above result holds if f and ψ have common poles in Theorem E? In this paper, we obtain the following result.

Theorem 3. *Let $\psi \not\equiv 0$ be a meromorphic function in a domain D and $k \in \mathbf{N}$, and let F be the family of meromorphic functions in D. If, for each $f \in \mathcal{F}$, f and $f^{(k)} - \psi$ have no zeros, and the poles of ψ are of multiplicity less than $k+1$ whenever f and ψ have common poles, then F is normal in D.*

Remark 2. The following example shows the condition that the poles of ψ are of multiplicity less than $k+1$ whenever f and ψ have common poles in Theorem 3 is necessary, and the number $k+1$ is sharp.

Example 1. Let $\mathcal{F} = \left\{ f_n(z) : f_n(z) = \frac{1}{nz},\ n = 1, 2, \cdots, \right\}$, $\psi(z) = \frac{1}{z^2}$, and $D = \{ z : |z| < 1 \}$. Obviously, $f_n(z) \neq 0$, $f_n'(z) - \psi(z) = -\frac{1}{nz^2} - \frac{1}{z^2} \neq 0$, $f_n(z)$ and $\psi(z)$ have the same pole $z = 0$, and the pole of $\psi(z)$ is of multiplicity 2. But it is easy to see that F is not normal in D.

2. Some lemmas

To prove our results, we need the following lemma, which is the well-known Zalcman's lemma.

Lemma 1. ([15]) *Let F be a family of functions meromorphic in a domain D. If F is not normal at $z_0 \in D$, then there exists a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

*locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on **C**.*

3. Proof of theorems

Proof of Theorem 1. Suppose that F is not normal at point $z_0 \in D$. Then by Lemma 1, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant function $g(\zeta)$.

We claim that $g^{(k)}(\zeta) \neq 0$.

Indeed, suppose that there exists a point ζ_0 such that $g^{(k)}(\zeta_0) = 0$. Since

$$g^{(k)}(\zeta) - \rho_n^k a = \rho_n^k (f_n^{(k)}(z_n + \rho_n\zeta) - a) \to g^{(k)}(\zeta),$$

by Hurwitz's theorem, there exist $\zeta_n, \zeta_n \to \zeta_0$, such that $f_n^{(k)}(z_n + \rho_n\zeta_n) = a$ (for n sufficiently large). It follows from the hypotheses on F that $f_n(z_n + \rho_n\zeta_n) = a$ or $f_n(z_n + \rho_n\zeta_n) = b$. Thus

$$g(\zeta_0) = a, \ or \ \ b \tag{1}$$

On the other hand,

$$g^{(k)}(\zeta) - \rho_n^k c = \rho_n^k (f_n^{(k)}(z_n + \rho_n\zeta) - c) \to g^{(k)}(\zeta).$$

Then using the same argument as the above, we deduce that

$$g(\zeta_0) = c \tag{2}$$

which contradicts (1).

Now we prove that $g(\zeta) \neq a, b$ and c. Suppose there exists $\zeta_1 \in \boldsymbol{C}$ such that $g(\zeta_1) = a$. Then by Hurwitz's theorem, there exist $\zeta_n, \zeta_n \to \zeta_1$ and

$$g_n(\zeta_n) = f_n\left(z_n + \rho_n\zeta_n\right) = a,$$

for sufficiently large n. Since f_n and $f_n^{(k)}$ share the set S_1, we have $f_n^{(k)}(z_n + \rho_n\zeta_n) = a$ or b, and then

$$g^{(k)}(\zeta_1) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \rho_n^k f_n^{(k)}\left(z_n + \rho_n\zeta_n\right) = 0,$$

a contradiction. Thus $g(\zeta) \neq a$. Similarly, we have $g(\zeta) \neq b$ and c. By Nevanlinna second fundamental theorem, $g(\zeta)$ must be a constant, a contradiction. This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose that F is not normal at point $z_0 \in D$. Then by Lemma 1, there exist a sequence of functions $f_n \in F$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant function $g(\zeta)$.

Suppose that $g(\zeta_0) = a_{i_0} (1 \le i_0 \le 3)$. Hurwitz's theorem implies the existence of a sequence $\zeta_n \to \zeta_0$ with

$$g_n(\zeta_n) = f_n(z_n + \rho_n\zeta_n) = a_{i_0}.$$

Since $f_n = a_{i_0} \Rightarrow f_n' = a_{i_0}$, we have $f_n'(z_n + \rho_n\zeta_n) = a$. Then

$$g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) = \lim_{n\to\infty} \rho_n f_n'(z_n + \rho_n\zeta_n) = 0,$$

and hence the zeros of $g(\zeta) - a_{i_0}$ are of multiplicity at least 2.

Without loss of generality, we assume that $a_1 \ne 0$ and $a_2 \ne 0$. Next we prove that $g(\zeta) \ne a_i (i = 1, 2)$. Suppose that ζ_0 is a a_1-point of $g(\zeta)$ of multiplicity $k(k \ge 2)$, then $g^{(k)}(\zeta_0) \ne 0$. Thus there exists a positive number δ, such that

$$g(\zeta) \ne a_1, \quad g'(\zeta) \ne 0, \quad g^{(k)}(\zeta) \ne 0, \tag{3}$$

on $D_\delta^0 = \{\zeta : 0 < |\zeta - \zeta_0| < \delta\}$. Since ζ_0 is a a_1-point of $g(\zeta)$ of multiplicity k, by Rouché theorem, there exists $\{\zeta_n^{(i)}\}$ $(i = 1, 2, \cdots, k)$ on $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$ such that $g_n(\zeta_n^{(i)}) - a_1 = 0$. Since

$$g_n'(\zeta_n^{(i)}) = \rho_n f_n'(z_n + \rho_n\zeta_n^{(i)}) = \rho_n a_1 \ne 0 \ (i = 1,2,\cdots,k),$$

so each $\zeta_n^{(i)}$ is a simple zero of $g_n(\zeta) - a_1$, that is,

$$g_n(\zeta_n^{(1)}) = g(\zeta_n^{(2)}) = \cdots = g_n(\zeta_n^{(k)}) = a_1(\zeta_n^{(i)} \ne \zeta_n^{(j)}, i \ne j).$$

On the other hand

$$\lim_{n\to\infty} g_n'(\zeta_n^{(i)}) = \lim_{n\leftarrow\infty} \rho_n a = 0,$$

then, from (3), we obtain

$$\lim_{n\to\infty} \zeta_n^{(i)} = \zeta_0 (i = 1, 2, \cdots, k).$$

Note that (3) and $g_n'(\zeta) - \rho_n a_1$ has k zeros $\zeta_n^{(1)}, \zeta_n^{(2)}, \cdots, \zeta_n^{(k)}$ in $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$, then ζ_0 is a zero of $g'(\zeta)$ of multiplicity k, and thus $g^{(k)}(\zeta_0) = 0$. This contradicts (3). Hence $g(\zeta) \neq a_1$. Similarly, $g(\zeta) \neq a_2$. By Nevanlinna second fundamental theorem, we arrive at a contradiction. This completes the proof of Theorem 2.

Remark 3. Some idea in the proof of Theorem 2 has its roots in Pang [4].

Proof of Theorem 3. Suppose $\psi(z_0) \neq \infty (z_0 \in D)$. This means that f and ψ have no common poles in D. By Theorem E, F is normal at z_0.

If $\psi(z_0) = \infty$, then there exists a positive number δ, such that $\psi(z)$ has no poles on $\{z : 0 < |z - z_0| < \delta\}$. Thus F is normal on $\{z : 0 < |z - z_0| < \delta\}$. Hence, for each function sequence $f_n \in F$, there exists a subsequence of $f_n(z)$ (without loss of generality, we still denote by $f_n(z)$), such that

$$f_n(z) \to f_0(z),$$

locally uniformly with respect to the spherical metric to $f_0(z)$ on $\{z : 0 < |z - z_0| < \delta\}$.

We consider two cases.

Case 1. $f_0(z) \not\equiv 0$.

Since $f_n(z) \neq 0$, by Hurwitz's theorem, $f_0(z)$ has no zeros on $\{z : 0 < |z - z_0| < \delta\}$. Then there exists a positive number m such that

$$\min_{0\le\theta<2\pi} \left| f_0\left(z_0 + \frac{\delta}{2} e^{i\theta} \right) \right| > m.$$

Thus there exists a positive integer N, such that

$$\min_{0\le\theta<2\pi} \left| f_n\left(z_0 + \frac{\delta}{2} e^{i\theta} \right) \right| > \frac{m}{2},$$

for $n \geq N$. Note that $f_n(z) \neq 0$ on D, by the minimum modulus theorem, we have

$$\min_{|z-z_0| \leq \frac{\delta}{2}} |f_n(z)| > \frac{m}{2}.$$

Thus F is normal at z_0.

Case 2. $f_0(z) \equiv 0$.

Then $f_n^{(k)}/\psi$ and $(f_n^{(k)}/\psi)'$ converges locally uniformly to 0 on $\{z : 0 < |z - z_0| < \delta\}$, so we have

$$\left| n\left(\frac{\delta}{2}, z_0, \frac{f_n^{(k)}}{\psi} - 1\right) - n\left(\frac{\delta}{2}, z_0, \frac{1}{\frac{f_n^{(k)}}{\psi} - 1}\right) \right| = \left| \frac{1}{2\pi i} \int_{|z-z_0| = \frac{\delta}{2}} \frac{\left(\frac{f_n^{(k)}}{\psi}\right)'}{\frac{f_n^{(k)}}{\psi} - 1} dz \right| < 1$$

for sufficiently large n. Since the poles of ψ are of multiplicity less than $k+1$ whenever f_n and ψ have common poles, and note that $f_n^{(k)} \neq \psi$, then

$$\overline{n}\left(\frac{\delta}{2}, z_0, f_n\right) \leq n\left(\frac{\delta}{2}, z_0, \frac{f_n^{(k)}}{\psi} z - 1\right) = n\left(\frac{\delta}{2}, z_0, \frac{1}{\frac{f_n^{(k)}}{\psi} - 1}\right) = 0.$$

It shows that $f_n(z)$ is holomorphic on $\{z : |z - z_0| < \delta/2\}$ for sufficiently large n. Thus $f_n(z)$ converges locally uniformly to 0 on $\{z : |z - z_0| < \delta/2\}$, and hence F is normal at z_0. This completes the proof of Theorem 3.

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