BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **27** (2004) 77–85

On Strongly 0-Prime Ideals in Near-Rings

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Abstract. In this paper we introduce the notion of strongly 0-prime ideals in near-rings similar to the notion introduced in rings. We give some characterizations of a near-ring N whose unique maximal nil ideal $N_r(N)$ coincides with the set of all its nilpotent elements $N(N)$ by using its minimal strongly 0-prime ideals.

2000 Mathematics Subject Classification: 16Y30.

1. Introduction

Throughout this paper N stands for a near-ring with identity. We use $N_r(N)$ and $N(N)$ to represent the unique maximal nil ideal and the set of all nilpotent elements of N respectively. Observe that $N_r(N) = N(N)$ if and only if $N_r(N)$ is a completely semiprime ideal of N (i.e., $a^2 \in N_r(N)$ implies $a \in N_r(N)$ for $a \in N$).

An ideal P of N is 0-prime if for any two ideals A and B of N, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of N is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for any a, $b \in N$ [1]. An ideal P of N is said to be strongly 0-prime if P is 0-prime and N/P has no non-zero nil ideals. A near-ring N is said to be strongly 0-prime if the ideal $\{0\}$ is strongly 0-prime. We have shown that if $M = \{x, x^2, x^3, \cdots\}$ where x is not a nilpotent element of N, then there exists a strongly 0-prime ideal P of N such that $P \cap M = \varnothing$. An ideal P of a near-ring is minimal strongly 0-prime ideal if P is minimal among strongly 0-prime ideals of N. Observe that every completely prime ideal of N is strongly 0-prime and every strongly 0-prime ideal is 0-prime but the converses do not hold.

An ideal I of N is said to have the insertion of factors property (or) simply *IFP* if $xy \in I$ implies $xNy \subseteq I$ for x, $y \in N$. An ideal I of N has the strict *IFP* if $xy \in I$ implies $\langle x\rangle N \langle y\rangle \subseteq I$ for x, $y \in N$. Observe that every completely semiprime ideal of N has the *IFP*. In a ring, *IFP* implies strict *IFP* but in a near-ring *IFP* does not imply strict *IFP*.

Recently, Kim and Kwak [3] characterized 2-primal rings in terms of their minimal prime ideals. Hong and Kwak [2] characterized a ring satisfying $N_r(R) = N(R)$ in terms of its minimal strongly prime ideals. So, in this paper we give some characterizations of a near-ring N whose unique maximal nil ideal $N_r(N)$ coincides with the set of all its nilpotent elements $N(N)$ by using its minimal strongly 0-prime ideals. For the basic definition and terminology we refer to [4].

Example 1.1. Consider the near-ring $(N, +, \cdot)$ defined on the Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$ where $\cdot$ is defined as follows (as per scheme 2, p.408 [4]).

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Clearly $\{0, a\}$ is a strongly 0-prime ideal, since the ideals are $\{0\}$, $\{0, a\}$ and $\{0, a, b, c\}$.

Let N be a near-ring and let $(m)\operatorname{Spec}(N)$ be the set of all (minimal) strongly 0-prime ideals of N. For $P \in \operatorname{Spec}(N)$, we put

$$O(P) = \{a \in N \mid aN\langle b\rangle = 0 \quad \text{for some } b \in N \setminus P\}.$$

$$\overline{O}(P) = \{a \in N \mid a^m \in O(P) \quad \text{for some positive integer } m\}.$$

$$O_p = \{a \in N \mid ab = 0 \quad \text{for some } b \in N \setminus P\}$$

$$\overline{O}_p = \{a \in N \mid a^m \in O_p \quad \text{for some positive integer } m\}.$$

$$N(P) = \{a \in N \mid aN\langle b\rangle \subseteq N_r(N) \quad \text{for some } b \in N \setminus P\}.$$

$$\overline{N}(P) = \{a \in N \mid a^m \in N(P) \quad \text{for some positive integer } m\}.$$

$$N_p = \{a \in N \mid ab \in N_r(N) \quad \text{for some } b \in N \setminus P\}.$$

$$\overline{N}_p = \{a \in N \mid a^m \in N_p \quad \text{for some positive integer } m\}.$$

Hong and Kwak [2] have defined $O(P)$ in a ring R as $O(P) = \{a \in R \mid aRb = 0$ for some $b \in R \setminus P\}$. But we have defined $O(P) = \{a \in R \mid aR\langle b\rangle = 0$ for some $b \in R \setminus P\}$. These two definitions coincide in rings. Suppose $a \in O(P)$. Then $aRb = 0$ for some $b \in R \setminus P$ implies $b \in (0: aR)_r = \{x \in R \mid aRx = 0\}$, which is an ideal if R is a ring. Thus $aRb = 0$ implies $aR\langle b\rangle = 0$.

We have shown that $O(P)$ and $N(P)$ are ideals of N and they are subsets of P. Clearly $O(P) \subseteq O_p \subseteq \overline{O}_p$, $N(P) \subseteq N_p \subseteq \overline{N}_p$, $\overline{O}(P) \subseteq \overline{O}_p$ and $\overline{N}(P) \subseteq \overline{N}_p$. If P is a completely prime ideal of N, then $\overline{N}_p$ is a subset of P. For a reduced near-ring N, $O(P) = O_p$ and $\overline{O}(P) = \overline{O}_p = N(P) = N_p = \overline{N}(P) = \overline{N}_p$.

Lemma 1.2. *$O(P)$ and $N(P)$ are ideals of N for any strongly 0-prime ideal P of N.*

Proof. Let P be any strongly 0-prime ideal of N and let $a_1, a_2 \in O(P)$. Then $a_1N\langle b_1\rangle = 0$ for some $b_1 \in N \setminus P$ and $a_2N\langle b_2\rangle = 0$ for some $b_2 \in N \setminus P$. Since b_1, $b_2 \in N \setminus P$ and $N \setminus P$ is an m-system there exists $b_1' \in \langle b_1\rangle$ and $b_2' \in \langle b_2\rangle$ such that $b_1' b_2' \in N \setminus P$. Let $b_3 = b_1' b_2'$. For any $n \in N$ and $x \in \langle b_3\rangle$, $(a_1 - a_2)nx = 0$ implies $a_1 - a_2 \in O(P)$. Let $x \in O(P)$. Then $xN\langle b\rangle = 0$ for some $b \in N \setminus P$. Thus for n, $n', n_1 \in N$ and $b' \in \langle b\rangle$, we have $(n(n' + x) - nn')n_1b' = 0$ implies $n(n' + x) - nn' \in O(P)$ and $(xn)n_1b' = 0$ implies $xn \in O(P)$. Therefore $O(P)$ is an ideal of N. Similarly one can show that $N(P)$ is an ideal of N.

Lemma 1.3. *For a near-ring N and $P \in \mathrm{Spec}(N)$, we have the following:*

(i) *If $O_p(N_p)$ is an ideal of N for any strongly 0-prime ideal of N, then $O_p(N_p)$ is a completely semiprime ideal of N if and only if $O_p = \overline{O}_p(N_p = \overline{N}_p)$.*

(ii) *$O(P)(N(P))$ is a completely semiprime ideal of N if and only if $O(P) = \overline{O}(P)(N(P) = \overline{N}(P))$.*

Proof.
(i) Let P be any strongly 0-prime ideal of N. Suppose that O_p is a completely semiprime ideal of N. Let $a \in \overline{O}_p$. Then $a^m b = 0$ for some positive integer m and for some $b \in N \setminus P$. Thus $a^m \in O_p$ and this implies $a \in O_p$ as O_P is completely semiprime. Therefore $\overline{O}_P \subseteq O_P$ and hence $\overline{O}_P = O_P$. The converse is obvious. Proof of part (ii) is similar to that of (i).

Theorem 1.4. *If* $M = \{x, x^2, x^3, \cdots\}$ *where x is not a nilpotent element of N, then there exist a strongly* 0*-prime ideal P of N such that* $P \cap M = \emptyset$.

Proof. Let $M = \{x, x^2, x^3, \cdots\}$ and $S = \{I \mid I \cap M = \emptyset,$ where I is an ideal of $N\}$. Then S is non-empty as $\{0\} \in S$. By Zorn's Lemma, S has a maximal element say P. We claim that P is strongly 0-prime. First we show that P is 0-prime. Suppose I_1 and I_2 are ideals of N such that $I_1 \supset P$ and $I_2 \supset P$. Let $a \in I_1 \cap M$ and $b \in I_2 \cap M$. Then we have $a = x^{n_1}$ and $b = x^{n_2}$ for some positive integers n_1, n_2. Therefore $ab = x^{n_1+n_2} \in I_1 I_2 \cap M$ implies $I_1 I_2 \cap M \neq \emptyset$ and hence $I_1 I_2 \subseteq P$. Therefore P is 0-prime. If I/P is a non-zero nil ideal of N/P, then $I \subseteq P$ and so $I \cap M \neq \emptyset$. Let $y \in I \cap M$. Then $y = x^k$ for some positive integer k. Since I/P is a nil ideal, $(x^k + P)^m = P$ for some positive integer m. Thus $x^{km} \in P$ which is a contradiction. Therefore P is a strongly 0-prime ideal of N such that $P \cap M = \emptyset$.

Lemma 1.5. *For a near-ring N,* $N_r(N) = \cap\{P \mid P$ *is a strongly* 0*-prime ideal of N}* $= \cap\{P \mid P$ *is a minimal strongly* 0*-prime ideal of N}.*

Proof. Suppose $N_r(N) \subseteq P$ for some $P \in \mathrm{Spec}(N)$. Then $N_r(N)/P$ is a non-zero nil ideal of N/P which is a contradiction that P is a strongly 0-prime ideal of N. Thus $N_r(N) \subseteq P$ for all strongly 0-prime ideals P of N and so $N_r(N) \subseteq \cap\{P \mid P$ is a strongly 0-prime ideal of $N\}$. Let $x \in \cap\{P \mid P$ is a strongly 0-prime ideal of $N\}$. If $x^k \neq 0$ for any positive integer k and if $M = \{x, x^2, x^3, \cdots\}$ then by Theorem 1.4, there exists a strongly 0-prime ideal P such that $P \cap M = \emptyset$. Thus $x \notin P$ which is a contradiction. Therefore $x^k = 0$ for some positive integer k. So $x \in N_r(N)$. The other equality is obvious.

Lemma 1.6. *For a near-ring, we have the following:*

(i) $N(N) \subseteq \bigcap_{P \in \mathrm{Spec}(N)} \overline{O}(P) \subseteq \bigcap_{Q \in m\,\mathrm{Spec}(N)} \overline{O}(Q)$.

(ii) $N_r(N) \subseteq \bigcap_{P \in \mathrm{Spec}(N)} N(P) = \bigcap_{Q \in m\,\mathrm{Spec}(N)} N(Q)$.

Proof.

(i) Let $a \in N(N)$. Then $a^n = 0$ for some positive integer n. Let P be any strongly 0-prime ideal and let $b \in N \setminus P$. Since $a^n = 0$, $a^n N\langle b\rangle = 0$. Thus $a^n \in O(P)$

and hence $a \in \overline{O}(P)$. Therefore $a \in \bigcap_{P \in \text{Spec}(N)} \overline{O}(P)$. The other inclusion is obvious.

(ii) Let $a \in N_r(N)$. Let P be any strongly 0-prime ideal of N. Then $aN\langle b \rangle \subseteq N_r(N)$ for any $b \in N \setminus P$ which implies $a \in N(P)$. Thus $a \in \bigcap_{P \in \text{Spec}(N)} N(P)$. But $\bigcap_{P \in \text{Spec}(N)} N(P) \subseteq \bigcap_{Q \in \text{Spec}(N)} N(Q)$ always. Since $N(Q) \subseteq Q$ for each $Q \in m\,\text{Spec}(N)$, $\bigcap_{Q \in m\,\text{Spec}(N)} N(Q) \subseteq N_r(N)$.

2. Strongly 0-prime ideals

Now we prove our main Theorem.

Theorem 2.1. *For a near-ring N, the following statements are equivalent.*

(i) $N_r(N) = N(N)$.

(ii) $N_r(N)$ *is a completely semiprime ideal of N.*

(iii) $N(P)$ *is a completely semiprime ideal of N for each* $P \in m\,\text{Spec}(N)$.

(iv) $\overline{N}_p = \overline{N}(P) = N(P)$ *for each* $P \in m\,\text{Spec}(N)$.

(v) $N(P) = N_p$ *for each* $P \in m\,\text{Spec}(N)$.

(vi) $N_p \subseteq P$ *for each* $P \in m\,\text{Spec}(N)$.

(vii) $N_{P/N_r(N)} \subseteq P/N_r(N)$ *for each* $P \in m\,\text{Spec}(N)$.

Proof.

(i) $\Rightarrow$ (ii) Since $N_r(N) = N(N)$ for any x in N, $x^2 \in N_r(N)$ implies x^2 is nilpotent and hence $x \in N(N) = N_r(N)$. Therefore $N_r(N)$ is a completely semiprime ideal of N.

(ii)$\Rightarrow$(iii) Let P be a minimal strongly 0-prime ideal of N. Let $x \in N$ be such that $x^2 \in N(P)$. Then $x^2 N\langle b \rangle \subseteq N_r(N)$ for some $b \in N \setminus P$. Since $N_r(N)$ is a completely semiprime ideal, it has the *IFP*. So $xNxN\langle b \rangle \subseteq N_r(N)$ which implies $xN\langle b \rangle \subseteq N_r(N)$. Thus $x \in N(P)$ and hence $N(P)$ is completely semiprime.

(iii)$\Rightarrow$(i) Let $a \in N(N)$. Then $a^n = 0$ for some positive integer n. If $a \notin N_r(N)$, then there exists a minimal strongly 0-prime ideal P of N such that $a \notin P$. Since $N(P)$ is a completely semiprime ideal, $a^n = 0 \in N(P)$ implies $a \in N(P) \subseteq P$, a contradiction. So $a \in N_r(N)$.

(ii)⇒(iv) Let P be a minimal strongly 0-prime ideal of N and let $a \in \overline{N}_p$ for some $a \in N$. Then $a^n \in N_p$ for some positive integer n. Thus $a^n b \in N_r(N)$ for some $b \in N \setminus P$. Since $N_r(N)$ is a completely semiprime ideal of N, it has the *IFP*. So we have $(ab)^n \in N_r(N)$ implies $ab \in N_r(N)$ and hence $a \in N(P)$. Thus $\overline{N}_p \subseteq N(P)$. But $N(P) \subseteq N_p \subseteq \overline{N}_p$ and $\overline{N}(P) = \overline{N}_p$. Therefore $\overline{N}_p = \overline{N}(P) = N(P)$ for each $P \in m\,\mathrm{Spec}\,(N)$.

(iv)⇒(v)⇒(vi) These are obvious.

(vi)⇒(vii) Let P be a minimal strongly 0-prime ideal of N. Let $\overline{N} = N / N_r(N)$ and $\overline{P} = P / N_r(N)$. Let $\overline{a} = a + N_r(N) \in N_{\overline{p}}$ for some $a \in N$. Then there exists $\overline{b} \in \overline{N} \setminus \overline{P}$ such that $\overline{a}\,\overline{b} \in N_r(\overline{N}) = \overline{0}$, which implies $ab \in N_r(N)$ and so $a \in N_p \subseteq P$. Thus $\overline{a} \in \overline{P}$ and hence $N_{\overline{p}} \subseteq \overline{P}$.

(vii)⇒(i) Suppose that $\overline{N} = N / N_r(N)$ is not reduced. Then there exists $\overline{a} \in \overline{N}$ such that $\overline{a}^2 = \overline{0}$ and $\overline{a} \neq \overline{0}$. Hence $a \notin N_r(N)$. So there exists some strongly 0-prime ideal P of N such that $a \notin \overline{P}$ and this implies $\overline{a} \in \overline{N} \setminus \overline{P}$. But $\overline{a}^2 = \overline{0}$ implies $\overline{a} \in N_{\overline{p}} \subseteq \overline{P}$, a contradiction. Therefore $N_r(N) = N(N)$.

Note that if R is a ring, then $N_r(R)$ has the *IFP* if and only if $N_r(R)$ is completely semiprime. Let us assume that $N_r(R)$ has *IFP* and let $x^2 \in N_r(R)$. Then $xRx \subseteq N_r(R)$ and hence $xRx \subseteq P$ for every strongly 0- prime ideal P of N. So $x \in P$ for every strongly 0-prime ideal P. Therefore $x \in N_r(R)$. Thus we have the following Corollary.

Corollary 2.2. [2, Theorem 8] *For a ring R, the following statements are equivalent.*

(i) $N_r(R) = N(R)$.

(ii) $N_r(R)$ *has the IFP.*

(iii) $N(P)$ *has the IFP for each* $P \in m\,\mathrm{Spec}\,(R)$.

(iv) $\overline{N}_p = \overline{N}(P) = N(P)$ *for each* $P \in m\,\mathrm{Spec}\,(R)$.

(v) $N(P) = N_p$ *for each* $P \in m\,\mathrm{Spec}\,(R)$.

(vi) $N_p \subseteq P$ *for each* $P \in m\,\mathrm{Spec}\,(R)$.

(vii) $N_{P / N_r(N)} \subseteq P / N_r(R)$ *for each* $P \in m\,\mathrm{Spec}\,(R)$.

Corollary 2.3. *For a near-ring N, assume that* $N_r(N) = N(N)$. *If* $P = N(P)$ *for each* $P \in \mathrm{Spec}\,(N)$, *then P is completely prime ideal of N.*

Proof. Let $N_r(N) = N(N)$ and $P = N(P)$ for each $P \in \text{Spec}(N)$. Let $ab \in P$ for $a, b \in N$. Since $N(P)$ is a completely semiprime ideal of N, we have $\langle a \rangle \langle b \rangle \subseteq P$ and hence $a \in P$ or $b \in P$. Therefore P is completely prime.

Theorem 2.4. *For a near-ring N, assume that* $N_r(N) = N(N)$. *Then for each* $P \in \text{Spec}(N)$, *the following statements are equivalent.*

(i) $P \in m\,\text{Spec}(N)$.

(ii) $N(P) = P$.

Proof.

(i)⇒(ii) Let $P \in m\,\text{Spec}(N)$ and $a \in P$. Suppose $a \notin N(P)$. Let $S = \{a, a^2, a^3, \cdots\}$. If $0 \in S$, then $a^k = 0$ for some positive integer k and hence $a \in N(N) = N_r(N)$, which implies $a \in N(P)$, a contradiction. So $0 \notin S$. Let $L = N \setminus P$ and let $T = \{a^{t_0} b_1 a^{t_1} b_2 \cdots b_n a^{t_n} \neq 0 \mid b_i \in L, t_i \in \{0\} \cup Z^+$, where Z^+ is the set of all positive integers$\}$. Then $L \subseteq T$. Let $M = S \cup T$. Let us show that M is an m-system. If $x, y \in S$, then $xay \in S$. Let $x \in S$, $y \in T$ with $x = a^s$, $y = a^{t_0} b_1 a^{t_1} b_2 \cdots b_n a^{t_n}$. If $xay \neq 0$, then $xay \in T$. Suppose $xay = 0$. Since $b_1, b_2 \in L$, there exists $b_1' \in \langle b_1 \rangle$ and $b_2' \in \langle b_2 \rangle$ such that $b_1' b_2' \in L$. Since $b_1' b_2', b_3 \in L$, there exists $b_{12}' \in \langle b_1' b_2' \rangle \subseteq \langle \langle b_1 \rangle \langle b_2 \rangle \rangle$ and $b_3' \in \langle b_3 \rangle$ such that $b_{12}' b_3' \in L$. Continuing this process we get $b_{123\cdots n-2}' b_{n-1}', b_n \in L$. Then there exists $b_{123\cdots n-1}' \in \langle b_{123\cdots n-2}' b_{n-1}' \rangle \subseteq \langle \cdots \langle \langle \langle b_1 \rangle \langle b_2 \rangle \langle b_3 \rangle \rangle \cdots \langle b_{n-1} \rangle \rangle$ and $b_n' \in \langle b_n \rangle$ such that $w = b_{123\cdots n-1}' b_n' \in L$. Since $xay = 0, xay \in N_r(N)$. Thus $a^s a a^{t_0} b_1 a^{t_1} b_2 \cdots b_n a^{t_n} \in N_r(N)$. Since $N_r(N) = N(N)$, $N_r(N)$ is a completely semiprime ideal of N and hence $b_1 b_2 \cdots b_n a^{1+s+t_0+\cdots t_n} \in N_r(N)$. Choose $m = 1 + s + t_0 + \cdots + t_n$. Then $b_1 b_2 \cdots b_n a^m \in N_r(N)$. Since $N_r(N)$ has the *IFP*, $\langle b_1 \rangle \langle b_2 \rangle \cdots \langle b_n \rangle \langle a^m \rangle \subseteq N_r(N)$. This implies $\langle \langle \langle b_1 \rangle \langle b_2 \rangle \langle b_3 \rangle \rangle \cdots \langle b_n \rangle \langle a^m \rangle \subseteq N_r(N)$. Continuing this process, we get $\langle \cdots \langle \langle \langle b_1 \rangle \langle b_2 \rangle \rangle \langle b_3 \rangle \rangle \cdots \langle b_{n-1} \rangle \rangle \langle b_n \rangle \langle a^m \rangle \subseteq N_r(N)$ and so $b_{123\cdots n-1}' b_n' a^m \in N_r(N)$. Hence $wa^m \in N_r(N)$ where $w = b_{123\cdots n-1}' b_n'$. Since $N_r(N)$ is a completely semiprime ideal, $(aw)^m \in N_r(N)$ and hence $aw \in N_r(N)$. Thus $a \in N_p = N(P)$, which is a contradiction.

Similarly, one can show that if $x, y \in T$ then $xay \neq 0$ and $xay \in T$. This shows that M is an m-system that is disjoint from (0). Hence there is a 0-prime ideal Q that is disjoint from M such that $a \notin Q$ and $Q \subseteq P$. Now we claim that Q is strongly

0-prime. Suppose *I/Q* is a non-zero nil ideal of *N/Q*. Since $Q \subset I$, $I \cap M \neq \emptyset$. If $a^m \in I$ for some positive integer *m*, then $a^m + Q$ is a nilpotent element in *N/Q*. Thus $a^{mk} \in Q$ for some positive integer *k*, which is a contradiction. So we choose $x \in I \cap T$. Then $x \in T$ implies $0 \neq x^t \in T$ for any positive integer *t*. Since $x + Q$ is nilpotent in *N/Q*, $x^s \in Q$ for some positive integer *s*, which is again a contradiction. Therefore Q is a strongly 0-prime ideal of *N* such that $a \notin Q$, a contradiction. Hence $N(P) = P$.

(ii)⇒(i) If $Q \subseteq P$ for $Q \in m\,\mathrm{Spec}(N)$, then $N(P) \subseteq N(Q) \subseteq Q \subseteq P = N(P)$. Therefore $P \in m\,\mathrm{Spec}(N)$.

Corollary 2.5. [2, Theorem 12] *For a ring R, assume that* $N_r(R) = N(R)$. *Then for each* $P \in \mathrm{Spec}(R)$, *the following statements are equivalent.*

(i) $P \in m\,\mathrm{Spec}(R)$.

(ii) $N(P) = P$.

A right ideal *I* of a near-ring *N* is called right (left) symmetric if $xyz \in I$ implies $xzy \in I (yxz \in I)$. An ideal *I* of *N* is symmetric if it is both right and left symmetric. An ideal *I* of *N* is called semi-symmetric if $x_1, x_2, \cdots x_n \in I$ implies $\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_n \rangle \subseteq I$.

Theorem 2.6. *For a near-ring N, the following statements are equivalent.*

(i) $N_r(N) = N(N)$.

(ii) *P is a completely prime ideal of N for each* $P \in m\,\mathrm{Spec}(N)$.

(iii) *P is a completely semiprime ideal of N for each* $P \in m\,\mathrm{Spec}(N)$.

(iv) *P has the strict IFP for each* $P \in m\,\mathrm{Spec}(N)$.

(v) *P is a symmetric ideal of N for each* $P \in m\,\mathrm{Spec}(N)$.

(vi) *P is a semi-symmetric ideal of N for each* $P \in m\,\mathrm{Spec}(N)$.

(vii) $ab \in P$ *implies* $bNa \subseteq P$ *for a,* $b \in N$ *and* $P \in m\,\mathrm{Spec}(N)$.

Proof.

(i)⇒(ii). It follows from Theorem 2.4 and Corollary 2.3.

(ii)⇒(iii)⇒(iv). These are obvious.

(iv)⇒(i). It follows from replacing $N(P)$ by *P* in the proof of (c)⇒(a) of Theorem 2.1.

(ii)⇒(v)⇒(vi)⇒(vii) and (vii)⇒(ii) are trivial.

Corollary 2.7. [2, Corollary 13] *For a ring R, the following statements are equivalent.*

(i) $N_r(R) = N(R)$.

(ii) *P is a completely prime ideal of R for each* $P \in m\,\mathrm{Spec}(R)$.

(iii) *P is a completely semiprime ideal of R for each* $P \in m\,\mathrm{Spec}(R)$.

(iv) *P has the IFP for each* $P \in m\,\mathrm{Spec}(R)$.

(v) *P is a symmetric ideal of R for each* $P \in m\,\mathrm{Spec}(R)$.

(vi) $xy \in P$ *implies* $yRx \subseteq P$ *for* $x, y \in R$ *and* $P \in m\,\mathrm{Spec}(R)$.

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