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Fractional Derivative of the Multivariable Polynomials

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Abstract. Motivated by several earlier works we establish a fractional derivative of the multivariable *H*-function [5], associated with a general class of multivariable polynomials [6], and the generalized Lauricella functions [1]. Certain interesting special cases have also been discussed.

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1. Introduction

The *H*-function of several complex variables introduced and studied by Srivastava and Panda [5] is an extansion of the multivariable *G*-function and includes Fox's *H*-function, Meijer's *G*-function of one and two variables, the generalized Lauricella functions [1], Appell functions, etc. In this note we have derive a fractional derivative of *H*-function of several complex variables [5], associated with a general class of multivariable polynomials [6] and the generalized Lauricella functions [1].

We start by giving the following definitions:

By Oldham and Spanier [2], and Srivastava and Goyal [3], the fractional derivative of a function $f(x)$ of complex order γ (or alternatively, a $-\gamma^{\text{th}}$ order fractional integral of $f(x)$ is defined as

$$ {}_aD_t^{-\gamma}\left[f(t)\right] = \begin{cases} \dfrac{1}{\Gamma(-\gamma)}\displaystyle\int_a^t (t-x)^{-\gamma-1} f(x)\,dx & , \quad \operatorname{Re}(\gamma) < 0 \\ \dfrac{d^m}{dt^m}\ {}_aD_t^{\gamma-m}\ \{f(t)\} & , \quad 0 \le \operatorname{Re}(\gamma) < m, \end{cases} \tag{1.1} $$

when *m* is a positive integer.

The *H*-function of several complex variables is defined by Srivastava and Panda [5] in the following manner:

$$H[z_1,\cdots,z_r]$$

$$\equiv H^{0,\lambda:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\cdots;[B^{(r)},D^{(r)}]}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\left|\begin{array}{l} \left[(a):\theta';\cdots;\theta^{(r)}\right]:\left[(b'):\Phi'\right];\cdots;\left[(b^{(r)}):\Phi^{(r)}\right] \\ \\ \left[(c):\psi';\cdots;\psi^{(r)}\right]:\left[(d'):\delta'\right];\cdots;\left[(d^{(r)}):\delta^{(r)}\right] \end{array}\right.\right] \quad (1.2)$$

$$= \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\cdots,\xi_r)\Phi_1(\xi_1)\cdots\Phi_r(\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}\,d\xi_1\cdots d\xi_r\ , \quad (1.3)$$

where $i=\sqrt{(-1)}$.

The convergence conditions and other details of the *H*-function of several complex variables $H[z_1,\cdots,z_r]$ are given by Srivastava, Gupta and Goyal [4].

For general class of multivariable polynomials (see [6]):

$$S_N^{M_1,\cdots,M_s}[w_1,\cdots,w_s]=\sum_{k_1,\cdots,k_s=0}^{M_1k_1+\cdots+M_sk_s}(-N)_{M_1k_1+\cdots+M_sk_s}B(N;k_1,\cdots,k_s)\frac{(w_1)^{k_1}}{k_1!}\cdots\frac{(w_s)^{k_s}}{k_s!}\ , \quad (1.4)$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $B(N;k_1,\cdots,k_s)$, $(N;k_{i'}\geq 0\,,\ i'=1,\cdots,s)$ are arbitrary constants, real or complex.

The following result [7] will be required in the sequel:

if $\lambda\geq 0,\ 0<x<1,\ \mathrm{Re}\,(1+p)>0,\ \mathrm{Re}\,(q)>-1,\ \lambda_i>0$ and $\Delta_i>0$, or $\Delta_i=0$ and $|z_i|<\sigma_i\,,\ i=1,2,\cdots,r$ then

$$x^{\lambda}F\begin{pmatrix} z_1x^{\lambda_1} \\ \vdots \\ z_rx^{\lambda_r}\end{pmatrix}=\sum_{M=0}^{\infty}\frac{(1+p+q+2M)(-\lambda)_M(1+p)_\lambda}{M!(1+p+q+M)_{\lambda+1}}F_M[z_1,\cdots,z_r]$$

$$\cdot\ {}_2F_1\left[\begin{array}{ll} -M,1+p+q+M & ; \\ & x \\ 1+p & ; \end{array}\right] \quad (1.5)$$

where

$$F_M\left[z_1,\cdots,z_r\right]$$

$$= F^{E+2:U';\cdots;U^{(r)}}_{P+2:V',\cdots,V^{(r)}}\left[\begin{matrix}[(e):\eta';\cdots;\eta^{(r)}],\,[1+p+\lambda:\lambda_1;\cdots;\lambda_r],\,[\lambda+1\\ [(g):\xi';\cdots;\xi^{(r)}],[2+p+q+M+\lambda:\lambda_1;\cdots;\lambda_r],\,[\lambda-M+1\end{matrix}\right.$$

$$\left.\begin{matrix}:\lambda_1;\cdots;\lambda_r]:[(w'):x'];\cdots;[(w^{(r)}:x^{(r)}]\;;\\ :\lambda_1;\cdots;\lambda_r]:[(v'):t'];\cdots;[(v^{(r)}):t^{(r)})]\;;\end{matrix}\; z_1,\cdots,z_r\right], \tag{1.6}$$

where $M \geq 0$.

For the sake of brevity, we use the following notation:

$$F^{E:U';\cdots;U^{(r)}}_{P:V';\cdots;V^{(r)}}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{pmatrix} \equiv F\begin{pmatrix}\gamma_1\\ \vdots\\ \lambda_r\end{pmatrix} \tag{1.7}$$

The special case of fractional derivative [2] is:

$$D_t^{\gamma}\left[t^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\gamma+1))}\, t^{\mu-\gamma} \;,\ \mathrm{Re}\,(\mu) < -1 \tag{1.8}$$

2. Main result

The fractional derivative to be evaluated here is:

$$D_t^{\gamma}\left\{ (t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho}\; F\begin{pmatrix}\tau_1\{\eta(y-t)\}^{\sigma_1}\\ \vdots\\ \tau_r\{\eta(y-t)\}^{\sigma_r}\end{pmatrix} S_N^{M_1,\cdots,M_s}\begin{bmatrix}(t-x)^{a_1}(y-t))^{b_1}\\ \vdots\\ (t-x)^{a_s}(y-t)^{b_s}\end{bmatrix}\right.$$

$$\left.\cdot\; H\begin{bmatrix}z_1\{t(t-x)\}^{\sigma_1}\{t(y-t)\}^{\rho_1}\\ \vdots\\ z_r\{t(t-x)\}^{\sigma_r}\{t(y-t)\}^{\rho_r}\end{bmatrix}\right\}$$

$$= \sum_{\alpha,\beta=0}^{\infty} \sum_{K,M=0}^{\infty} \sum_{k_1,\cdots,k_s=0}^{M_1k_1+\cdots+M_sk_s\le N} \frac{(-N)_{M_1k_1+\cdots+M_sk_s}}{k_1!\cdots k_s!} B\left(N;k_1,\cdots,k_s\right)\Delta$$

$$\cdot\, H^{2,\lambda+1:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A+3,C+3:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left[\begin{array}{c} z_1(-x)^{\sigma_1} y^{\rho_1} t^{\rho_1+\sigma_1} \\ \vdots \\ z_r(-x)^{\sigma_r} y^{\rho_r} t^{\rho_r+\sigma_r} \end{array}\right| \begin{array}{l} (-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right), \end{array}$$

$$\left[(a):\theta';\cdots;\theta^{(r)}\right],\ \left(-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right),$$
$$\left(\beta-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right),\ \left[(c):\psi';\cdots;\psi^{(r)}\right],$$

$$\left.\begin{array}{l} \left(-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right):\left[(b'):\Phi'\right];\cdots;\left[(b^{(r)}):\Phi^{(r)}\right] \\ (\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r):\left[(d'):\delta'\right];\cdots;\left[(d^{(r)}:\delta^{(r)}\right] \end{array}\right], \quad (2.1)$$

where

$$\Delta = (-1)^{\alpha} \frac{(1+p+q+2M)(1+p+q+M)_K(-M)_K(-\sigma)_K(1+p)_\sigma}{K!M!(1+p+q+M)_{\sigma+1}(1+p)_K\Gamma(\alpha+1)\Gamma(\beta+1)}\eta^K$$

$$(-x)^{\sigma-\alpha+\sum_{i'=1}^{s} a_{i'}k_{i'}} \cdot y^{\rho+K-\beta+\sum_{i'=1}^{s} b_{i'}k_{i'}}\, t^{\alpha+\beta-\gamma}\, F_M\left[z_1,\cdots,z_r\right];$$

$$\sigma_i > 0,\ \rho_i > 0,\ i = 1,2,\cdots,r$$

and
$$\mathrm{Re}(\sigma) + \sum_{i=1}^{r}\sigma_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ \mathrm{Re}(\rho) + \sum_{i=1}^{r}\rho_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1.$$

3. Interesting special cases:

(I) With $\lambda = A = C = 0$, the multivariable H-function breaks into product of r Fox's H-functions and consequently there holds the following result:

$$D_t^{\gamma}\left\{ (t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho} F\left(\begin{array}{c} \tau_1\{\eta(y-t)\}^{\sigma_1} \\ \vdots \\ \tau_r\{\eta(y-t)\}^{\sigma_r} \end{array}\right) S_N^{M_1,\cdots,M_s}\left[\begin{array}{c} (t-x)^{a_1}(y-t)^{b_1} \\ \vdots \\ (t-x)^{a_s}(y-t)^{b_s} \end{array}\right]\right.$$

$$\cdot \prod_{i=1}^{r} H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}} \left[z_i \{t(t-x)\}^{\sigma_i} \{t(y-t)\}^{\rho_i} \middle| \begin{matrix} [b^{(i)} : \Phi^{(i)}] \\ [(d^{(i)}) : \delta^{(i)}] \end{matrix} \right] \Bigg\}$$

$$= \sum_{\alpha,\beta=0}^{\infty} \sum_{K,M=0}^{\infty} \sum_{k_1,\cdots,k_s=0}^{M_1k_1+\cdots+M_sk_s \le N} \frac{(-N)_{M_1k_1+\cdots+M_sk_s}}{k_1!\cdots k_s!} B\big(N;k_1,\cdots,k_s\big)\Delta$$

$$\cdot H_{3,3:[B',D'];\cdots;[B^{(r)},D^{(r)}]}^{2,1:(u',v');\cdots;(u^{(r)},v^{(r)})} \left[\begin{matrix} z_1(-x)^{\sigma_1} y^{\rho_1} t^{\rho_1+\sigma_1} \\ \vdots \\ z_r(-x)^{\sigma_r} y^{\rho_r} t^{\rho_r+\sigma_r} \end{matrix} \middle| \begin{matrix} (-\alpha-\beta : \rho_1+\sigma_1;\cdots;\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'} : \sigma_1;\cdots;\sigma_r\right), \end{matrix} \right.$$

$$\left(-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'} : \sigma_1;\cdots;\sigma_r\right), \quad \left(-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'} : \rho_1;\cdots;\rho_r\right):$$
$$\left(\beta-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'} : \rho_1;\cdots;\rho_r\right), (\gamma-\alpha-\beta : \rho_1+\sigma_1;\cdots;\rho_r+\sigma_r):$$

$$\left. \begin{matrix} [(b') : \Phi'];\cdots;[(b^{(r)}) : \Phi^{(r)}] \\ [(d') : \delta'];\cdots;[(d^{(r)}) : \delta^{(r)}] \end{matrix} \right], \tag{3.1}$$

valid under the conditions surrounding (2.1).

(II) If $\Phi^{(i)} = \delta^{(i)} = 1, (i = 1, 2, \cdots, r)$, we get the following:

$$D_t^{\gamma} \left\{ (t-x)^{\sigma} \eta^{\sigma} (y-t)^{\sigma+\rho} F \begin{pmatrix} \tau_1\{\eta(y-t)\}^{\sigma_1} \\ \vdots \\ \tau_r\{\eta(y-t)\}^{\sigma_r} \end{pmatrix} S_N^{M_1,\cdots,M_s} \begin{bmatrix} (t-x)^{a_1}(y-t)^{b_1} \\ \vdots \\ (t-x)^{a_s}(y-t)^{b_s} \end{bmatrix} \right.$$

$$\cdot \prod_{i=1}^{r} G_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}} \left[z_i \{t(t-x\}^{\sigma_i} \{t(y-t)\}^{\rho_i} \middle| \begin{matrix} (b^{(i)}) \\ (d^{(i)}) \end{matrix} \right] \Bigg\}$$

$$= \sum_{\alpha,\beta=0}^{\infty} \sum_{K,M=0}^{\infty} \sum_{k_1,\cdots,k_s=0}^{M_1k_1+\cdots+M_sk_s \le N} \frac{(-N)_{M_1k_1+\cdots+M_sk_s}}{k_1!\cdots k_s!} B\big(N;k_1,\cdots,k_s\big)\Delta$$

$$\cdot\, H^{2,1:(u',v');\cdots;(u^{(r)},v^{(r)})}_{3,3:[B',D'],\cdots;[B^{(r)},D^{(r)}]}\left[\begin{matrix} z_1(-x)^{\sigma_1}\, y^{\rho_1}\, t^{\rho_1+\sigma_1} \\ \vdots \\ z_r(-x)^{\sigma_r}\, y^{\rho_r}\, t^{\rho_r+\sigma_r}\end{matrix}\right| \begin{matrix}\left(-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r\right), \\ \left(\alpha-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right),\end{matrix}$$

$$\begin{matrix}\left(-\sigma-\sum_{i'=1}^{s} a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right), \left(-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right) \\ \left(\beta-\rho-K-\sum_{i'=1}^{s} b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right), \left(\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r\right)\end{matrix}$$

$$\left.\begin{matrix}:(b');\cdots;(b^{(r)}) \\ :(d');\cdots;(d^{(r)})\end{matrix}\right], \tag{3.2}$$

valid under the conditions as obtainable from (2.1).

(III) Taking $B(N;k_1,\cdots,k_s)=I(\theta)$ in (1.4) where

$$I(\theta)=\frac{\prod_{j=1}^{E}(e_j)_{k_1\eta'_j+\cdots+k_s\eta_j^{(s)}}\prod_{j=1}^{U'}(u'_j)_{k_1x'_j}\cdots\prod_{j=1}^{U^{(s)}}(u_j^{(s)})_{k_sx_j^{(s)}}}{\prod_{j=1}^{P}(g_j)_{k_1\xi_j^1+\cdots+k_s\xi_j^{(s)}}\prod_{j=1}^{V'}(v'_j)_{k_1t'_j}\cdots\prod_{j=1}^{V^{(s)}}(v_j^{(s)})_{k_st_j^{(s)}}}$$

$S_N^{M_1,\cdots,M_s}\left[w_1,\cdots,w_s\right]$ reduces to the generalized Lauricella function of Srivastava and Daoust [1],

$$S_N^{M_1,\cdots,M_s}\left[w_1,\cdots,w_s\right]$$

$$=F^{1+E:U';\cdots;U^{(s)}}_{P:V';\cdots;V^{(s)}}\left[\begin{matrix}\left[-N:M_1;\cdots;M_s\right],\left[(e):\eta';\cdots;\eta^{(s)}\right]:\left[(w'):x'\right];\cdots; \\ \left[(g):\xi';\cdots;\xi^{(s)}\right]:\left[(v'):t'\right];\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}\left[(w^{(s)}:x^{(s)}\right]; \\ \left[(v^{(s)}):t^{(s)}\right];\end{matrix}\; w_1,\cdots,w_s\right]$$

and our main result (2.1) reduces to the following formula:

$$D_t^{\gamma}\left\{(t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho}\; F\begin{pmatrix}\tau_1\{\eta(y-t)\}^{\sigma_1}\\ \vdots \\ \tau_r\{\eta(y-t)\}^{\sigma_r}\end{pmatrix} F^{1+E:U';\cdots;U^{(s)}}_{P:V';\cdots;V^{(s)}}\left[\begin{matrix}[-N:M_1;\cdots;M_s], \\ \left[(g):\xi';\cdots;\xi^{(s)}\right]\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left[(e):\eta';\cdots;\eta^{(s)}\right]:\left[(w'):x'\right];\cdots;\left[(w^{(s)}):x^{(s)}\right]; \\ :\left[(v'):t'\right];\cdots;\left[(v^{(s)}):t^{(s)}\right]\;;\end{matrix}\quad \begin{matrix}(t-x)^{a_1}(y-t)^{b_1}\\ \vdots \\ (t-x)^{a_s}(y-t)^{b_s}\end{matrix}\right]$$

$$\cdot\, H\left[\begin{matrix}z_1\{t(t-x)\}^{\sigma_1}\{t(y-t)\}^{\rho_1}\\ \vdots \\ z_r\{t(t-x)\}^{\sigma_r}\{t(y-t)\}^{\rho_r}\end{matrix}\right]$$

$$=\sum_{\alpha,\beta=0}^{\infty}\;\sum_{K,M=0}^{\infty}\;\sum_{k_1,\cdots,k_s=0}^{M_1k_1+\cdots+M_sk_s\le N}\frac{(-N)_{M_1k_1+\cdots+M_sk_s}}{k_1!\cdots k_s!}\, I(\theta)\,\Delta$$

$$\cdot\; H^{2,\lambda+1:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A+3,C+3:[B',D'];\cdots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}z_1(-x)^{\sigma_1}y^{\rho_1}t^{\rho_1+\sigma_1}\\ \vdots \\ z_r(-x)^{\sigma_r}y^{\rho_r}t^{\rho_r+\sigma_r}\end{matrix}\right|\begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i'=1}^{s}a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right),\end{matrix}$$

$$\begin{matrix}\left[(a):\theta';\cdots;\theta^{(r)}\right],\;\left(-\sigma-\sum_{i'=1}^{s}a_{i'}k_{i'}:\sigma_1;\cdots;\sigma_r\right), \\ \left(\beta-\rho-K-\sum_{i'=1}^{s}b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right),\left[(c):\psi';\cdots;\psi^{(r)}\right],\end{matrix}$$

$$\left.\begin{matrix}\left(-\rho-K-\sum_{i'=1}^{s}b_{i'}k_{i'}:\rho_1;\cdots;\rho_r\right):\left[(b'):\Phi'\right];\cdots;\left[(b^{(r)}):\Phi^{(r)}\right] \\ (\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r):\left[(d'):\delta'\right];\cdots;\left[(d^{(r)}:\delta^{(r)}\right]\end{matrix}\right], \qquad (3.3)$$

Which holds true under the same conditions as given in (2.1).

(IV) Letting $M_i = 0, (i = 1, 2, \cdots, s)$ and $N \to 0$, the result in (2.1) reduces to the known result given in [8], after a little simplification.

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