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On LP-Sasakian Manifolds

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Abstract. The object of the present paper is to study LP-Sasakian manifolds satisfying certain conditions.

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1. Introduction

In 1989 K. Matsumoto [1] introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [2] introduced the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [3]; U.C. De, K. Matsumoto and A.A. Shaikh [4].

In [5] Yano and Sawaki defined and studied a tensor field W on a Riemannian manifold of dimension n which includes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ as special cases. This tensor field W is known as quasi-conformal curvature tensor. The quasi-conformally flat Riemannian manifold has been studied in [6]. The present paper deals with a study of LP-Sasakian manifolds satisfying certain conditions. After preliminaries, in section 3 we study an LP-Sasakian manifold satisfying the condition $R(X,Y)\cdot W = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y and it is shown that such a manifold is an Einstein manifold. Also in such a manifold we obtain a necessary and sufficient condition for the characteristic vector field ξ to be a harmonic vector field. Section 4 is devoted to the study of quasi-conformally recurrent LP-Sasakian manifolds. The last section deals with an LP-Sasakian manifold satisfying the condition $S(X,\xi)\cdot R = 0$ and it is proved that such a manifold is η-Einstein.

2. Preliminaries

An n-dimensional differentiable manifold M is called an LP-Sasakian manifold [1], [3] if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$
$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.3}$$
$$g(X,\xi) = \eta(X), \quad \nabla_X \xi = \phi X, \tag{2.4}$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric *g*.

It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.6}$$

$$\text{rank } \phi = n-1. \tag{2.7}$$

Again if we put

$$\Omega(X,Y) = g(X,\phi Y) \tag{2.8}$$

for any vector fields *X* and *Y*, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [1]. Also since the vector field η is closed in an LP-Sasakian manifold, we have [1] [4]

$$\text{(i)} \ (\nabla_X \eta)(Y) = \Omega(X,Y), \quad \text{(ii)} \ \Omega(X,\xi) = 0 \tag{2.9}$$

for any vector fields *X* and *Y*.

An LP-Sasakian manifold *M* is said to be η-Einstein if its Ricci tensor *S* is of the form

$$S(X,Y) = \alpha\, g(X,Y) + \beta\, \eta(X)\eta(Y) \tag{2.10}$$

for any vector fields *X*, *Y* where α, β are functions on *M*. Let *M* be an *n*-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then we have [3] [4]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.11}$$
$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.12}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$
$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.14}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{2.15}$$
$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\,\eta(X)\,\eta(Y) \tag{2.16}$$

for any vector fields *X*, *Y*, *Z* where $R(X,Y)Z$ is the Riemannian curvature tensor.

The quasi-conformal curvature tensor W on a manifold M of dimension n is defined by [5]

$$W(X,Y)Z = -(n-2)b\ C(X,Y)Z + [a + (n-2)b]\ \tilde{C}(X,Y)Z, \tag{2.17}$$

where a, b are arbitrary constants such that a and b are not zero simultaneously, C and $\tilde{C}$ are conformal curvature tensor and concircular curvature tensor respectively, given by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY\big] + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned} \tag{2.18}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{2.19}$$

Q is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Using (2.18) and (2.19) in (2.17) we get

$$\begin{aligned} W(X,Y)Z = a\,R(X,Y)Z + b\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY\big] - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right]\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.20}$$

The above results will be used in the next sections.

3. LP-Sasakian manifolds satisfying $R(X,Y)\cdot W = 0$

Let us consider an LP-Sasakian manifold (M^n, g) satisfying the condition

$$R(X,Y)\cdot W = 0. \tag{3.1}$$

Now, $$\begin{aligned} (R(X,Y)\cdot W)(U,V)Z = R(X,Y)W(U,V)Z - W(R(X,Y)U,V)Z \\ -W(U,R(X,Y)V)Z - W(U,V)R(X,Y)Z. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned} g\big(R(\xi,Y)W(U,V)Z,\xi) - g(W(R(\xi,Y)U,V)Z,\xi) - \\ g(W(U,R(\xi,Y)V)Z,\xi) - g(W(U,V)R(\xi,Y)Z,\xi) = 0. \end{aligned} \tag{3.3}$$

By virtue of (2.12) we obtain from (3.3)

$$\begin{aligned}&-\overline{W}(U,V,Z,Y)-\eta(Y)\eta(W(U,V)Z)-g(Y,U)\eta(W(\xi,V)Z)\\&+\eta(U)\eta(W(Y,V)Z)-g(Y,V)\eta(W(U,\xi)Z)+\eta(V)\eta(W(U,Y)Z)\\&-g(Y,Z)\eta(W(U,V)\xi)+\eta(Z)\eta(W(U,V)Y)=0,\end{aligned}\tag{3.4}$$

where $\overline{W}(U,V,Z,Y)=g(W(U,V)Z,Y)$.

From (2.20), it follows that

$$\eta(W(U,V)\xi)=0.\tag{3.5}$$

Using (3.5) in (3.4) we get

$$\begin{aligned}&-\overline{W}(U,V,Z,Y)-\eta(Y)\eta(W(U,V)Z)-g(Y,U)\eta(W(\xi,V)Z)\\&+\eta(U)\eta(W(Y,V)Z)-g(Y,V)\eta(W(U,\xi)Z)\\&+\eta(V)\eta(W(U,Y)Z)+\eta(Z)\eta(W(U,V)Y)=0.\end{aligned}\tag{3.6}$$

Let, $\{e_i : i=1,2,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $U=Y=e_i$ in (3.6) and taking summation for $1\le i\le n$ we get

$$\sum_{i=1}^{n}\varepsilon_i\,\overline{W}(e_i,V,Z,e_i)+(n-1)\eta(W(\xi,V)Z)-\eta(Z)\sum_{i=1}^{n}\varepsilon_i\eta(W(e_i,V)e_i)=0,\tag{3.7}$$

where $\varepsilon_i=g(e_i,e_i)$.

From (2.20), it follows that

$$\sum_{i=1}^{n}\varepsilon_i\,\overline{W}(e_i,V,Z,e_i)=[a+(n-2)b]\{S(V,Z)-\frac{r}{n}g(V,Z)\},\tag{3.8}$$

$$\sum_{i=1}^{n}\varepsilon_i\,\eta(W(e_i,V)e_i)=\big[a+(n-2)b\big]\left[\frac{r}{n}-(n-1)\right]\eta(V),\tag{3.9}$$

$$\begin{aligned}\eta(W(\xi,V)Z)=&-b\left[S(V,Z)-\left(\frac{r}{n-1}-1\right)g(V,Z)-\left(\frac{r}{n-1}-n\right)\eta(V)\eta(Z)\right]\\&+\big[a+(n-2)b\big]\left[\frac{r}{n(n-1)}-1\right]\{g(V,Z)+\eta(V)\eta(Z)\}.\end{aligned}\tag{3.10}$$

Using (3.8) - (3.10) in (3.7) we obtain

$$(a-b)S(V,Z) = \left[(n-1)\{a+(n-1)b\} - br\right]g(V,Z) + b[n(n-1)-r]\eta(V)\eta(Z). \tag{3.11}$$

From (3.11) we have

$$S(V,Z) = \frac{[(n-1)\{a+(n-1)b\} - br]}{a-b}\, g(V,Z) + \frac{b}{a-b}\,[n(n-1)-r]\eta(V)\eta(Z) \tag{3.12}$$

provided that $a-b \neq 0$. The relation (3.12) implies that the manifold is η-Einstein.

Hence we can state the following:

Theorem 3.1. *An LP-Sasakian manifold* $(M^n, g)\,(n>3)$ *satisfying the condition* $R(X,Y)\cdot W = 0$ *is an* η*-Einstein manifold provided that* $a-b \neq 0$.

If $a=b$, then (3.11) yields

$$r = n(n-1).$$

This leads to the following:

Corollary 3.1. *An LP-Sasakian manifold* $(M^n, g)\,(n>3)$ *satisfying the condition* $R(X,Y)\cdot W = 0$ *is of constant scalar curvature for* $a=b$.

Taking an orthonormal frame field and contracting (3.11) over *V* and *Z* we obtain

$$r = n(n-1) \quad \text{if} \quad a+(n-2)b \neq 0.$$

Using this value of r in (3.12) we get

$$S(V,Z) = (n-1)g(V,Z) \quad \text{if} \quad a-b \neq 0.$$

Thus we can state the following:

Theorem 3.2. *An LP-Sasakian manifold* $(M^n, g)\,(n>3)$ *satisfying the condition* $R(X,Y)\cdot W = 0$ *is an Einstein manifold provided that* $a-b\neq 0$ *and* $a+(n-2)b \neq 0$.

If $a+(n-2)b = 0$, then from (2.17) it follows that $W(X,Y)Z = -(n-2)bC(X,Y)Z$ and hence $R(X,Y)\cdot W = 0$ implies

$R(X,Y)\cdot C = 0$. Again, LP-Sasakian manifolds satisfying the condition $R(X,Y)\cdot C = 0$ has been studied in [4].
This leads to the following:

Corollary 3.2. *An LP-Sasakian manifold* $(M^n, g)\,(n > 3)$ *satisfying the condition* $R(X,Y)\cdot W = 0$ *is a space of constant curvature if* $a + (n-2)b = 0$.

Let us now consider an LP-Sasakian manifold $(M^n, g)\,(n > 3)$ satisfying the condition $R(X,Y)\cdot W = 0$ which is not an Einstein one. Then (3.11) holds good. Differentiating (3.11) covariantly along X and then using (2.9) (i) we get

$$(a-b)(\nabla_X S)(V,Z) = -bdr(X)\left[g(V,Z) + \eta(V)\eta(Z)\right] + b[n(n-1) - r]\left[\Omega(X,V)\eta(Z) + \Omega(X,Z)\eta(V)\right]. \tag{3.13}$$

Putting $X = Z = e_i$ in (3.13) and then taking summation for $1 \le i \le n$ we obtain by virtue of (2.9) (ii)

$$\frac{1}{2}(a+b)\,dr(V) = b\{[n(n-1) - r]\psi - dr(\xi)\}\,\eta(V), \tag{3.14}$$

where $\psi = \sum_{i=1}^{n} \varepsilon_i\, \Omega\,(e_i, e_i) = tr\cdot\phi$.

Replacing V by ξ in (3.14) we get

$$dr(\xi) = \frac{2b}{a-b}\left[r - n(n-1)\right]\psi, \text{ if } a - b \neq 0. \tag{3.15}$$

By virtue of (3.14) and (3.15) we obtain

$$dr(V) = \frac{2b}{a-b}\left[n(n-1) - r\right]\psi\,\eta(V), \text{ if } a + b \neq 0. \tag{3.16}$$

If r is constant then for $b \neq 0$, (3.16) yields either $r = n(n-1)$, or $\psi = 0$. If $r = n(n-1)$, then from (3.12), it follows that the manifold is Einstein. Hence if the manifold is not Einstein then we must have $\psi = 0$, which means that the vector field ξ is harmonic. Again, if $\psi = 0$, then from (3.16), it follows that r is constant.

Thus we can state the following:

Theorem 3.3. *Let* $(M^n, g)\,(n > 3)$ *be an LP-Sasakian manifold satisfying the condition* $R(X,Y)\cdot W = 0$ *which is not an Einstein one. Then the scalar curvature of the manifold is constant if and only if the timelike vector field* ξ *is harmonic provided that* $a \pm b \neq 0$ *and* $b \neq 0$.

In particular, if $a = b$ then we have the following:

Corollary 3.3. *Let* $(M^n, g)\,(n > 3)$ *be an LP-Sasakian manifold satisfying the condition* $R(X,Y)\cdot W = 0$ *which is not an Einstein one. If* $a = b$ *then the manifold is of constant scalar curvature.*

4. Quasi-conformally recurrent LP-Sasakian manifolds

A non-flat Riemannian manifold M is said to be quasi-conformally recurrent [7] if the quasi-conformal curvature tensor W satisfies the condition $\nabla W = A \otimes W$, where A is an everywhere non-zero 1-form. We now define a function f on M by $f^2 = g(W,W)$, where the metric g is extended to the inner product between the tensor fields in the standard fashion.

Then we know that $f(Yf) = f^2 A(Y)$. So from this we have

$$Yf = f\,A(Y) \quad (\text{because } f \neq 0). \tag{4.1}$$

From (4.1) we have

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore we get

$$\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) f = \left\{XA(Y) - YA(X) - A([X,Y])\right\} f.$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M by our assumption we obtain

$$dA(X,Y) = 0, \tag{4.2}$$

that is, the 1-form A is closed.

Now, from $(\nabla_X W)(U,V)Z = A(X)W(U,V)Z$, we get

$$(\nabla_U \nabla_V W)(X,Y)Z = \{UA(V) + A(U)A(V)\}W(X,Y)Z.$$

Hence from (4.2) we get

$$(R(X,Y)\cdot W)(U,V)Z = [2dA(X,Y)]W(U,V)Z = 0.$$

Therefore, for a quasi-conformally recurrent manifold, we have

$$R(X,Y)\cdot W = 0 \quad \text{for all } X, Y.$$

Thus we can state the following:

Theorem 4.1. *A quasi-conformally recurrent LP-Sasakian manifold* $(M^n, g)\,(n > 3)$ *is an Einstein manifold provided that* $a - b \neq 0$ *and* $a + (n-2)b \neq 0$.

Corollary 4.1. *A quasi-conformally recurrent LP-Sasakian manifold* $(M^n, g)\,(n > 3)$ *is of constant scalar curvature for* $a = b$ *and is a space of constant curvature if* $a + (n-2)b = 0$.

Since for a quasi-conformally symmetric LP-Sasakian manifold $(M^n, g)\,(n > 3)$, we have $(\nabla_U W)(X,Y)Z = 0$ which implies $R(X,Y)\cdot W = 0$, we can state the following:

Corollary 4.2. *A quasi-conformally symmetric LP-Sasakian manifold* $(M^n, g)\,(n > 3)$ *is an Einstein manifold provided that* $a - b \neq 0$ *and* $a + (n-2)b \neq 0$.

5. LP-Sasakian manifolds satisfying the condition $S(X,\xi)\cdot R = 0$

We now consider an LP-Sasakian manifold $(M^n, g)\,(n > 3)$ satisfying the condition

$$(S(X,\xi)\cdot R)(U,V)Z = 0. \tag{5.1}$$

By definition we have

$$\begin{aligned}(S(X,\xi)\cdot R)(U,V)Z &= ((X \wedge_S \xi)\cdot R)(U,V)Z \\ &= (X \wedge_S \xi)R(U,V)Z + R((X \wedge_S \xi)U,V)Z \\ &\quad + R(U,(X \wedge_S \xi)V)Z + R(U,V)(X \wedge_S \xi)Z,\end{aligned} \tag{5.2}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{5.3}$$

Using the definition of (5.3) in (5.2) we get by virtue of (2.15)

$$\begin{aligned}(S(X,\xi)\cdot R)(U,V)Z = (n-1)[&\eta(R(U,V)Z)X + \eta(U)\,R(X,V)Z \\ &+ \eta(V)\,R(U,X)Z + \eta(Z)\,R(U,V)X] \\ &- S(X,R(U,V)Z)\xi - S(X,U)\,R(\xi,V)Z \\ &- S(X,V)\,R(U,\xi)Z - S(X,Z)\,R(U,V)\xi.\end{aligned} \tag{5.4}$$

In view of (5.1) and (5.4) we have

$$\begin{aligned}(n-1)\big[\eta(R(U,V)Z)X &+ \eta(U)\,R(X,V)Z + \eta(V)\,R(U,X)Z + \eta(Z)\,R(U,V)X\big] \\ &- S(X,R(U,V)Z)\xi - S(X,U)\,R(\xi,V)Z \\ &- S(X,V)\,R(U,\xi)Z - S(X,Z)\,R(U,V)\xi = 0.\end{aligned} \tag{5.5}$$

Taking the inner product on both sides of (5.5) by ξ we obtain

$$\begin{aligned}(n-1)\big[&\eta(R(U,V)Z)\eta(X) + \eta(U)\eta(R(X,V)Z) + \eta(V)\eta(R(U,X)Z) \\ &+ \eta(Z)\,\eta(R(U,V)X)\big] + S(X,R(U,V)Z) - S(X,U)\eta(R(\xi,V)Z) \\ &- S(X,V)\eta(R(U,\xi)Z) - S(X,Z)\eta(R(U,V)\xi) = 0.\end{aligned} \tag{5.6}$$

Putting $U = Z = \xi$ in (5.6) and using (2.11) – (2.15) we get

$$S(X,V) = (1-n)\,g(X,V) + 2(1-n)\,\eta(X)\,\eta(V), \tag{5.7}$$

which means that the manifold is η-Einstein. This leads to the following:

Theorem 5.1. *An LP-Sasakian manifold* $(M^n, g)\,(n > 3)$ *satisfying the condition* $S(X,\xi)\cdot R = 0$ *is an* η*-Einstein manifold.*

Again, differentiating (5.7) covariantly along Y and using (2.9) we get

$$(\nabla_Y S)(X,V) = 2(1-n)\big[\Omega(X,Y)\,\eta(V) + \Omega(Y,V)\,\eta(X)\big]. \tag{5.8}$$

Taking an orthonormal frame field and contracting (5.8) over Y and V we obtain

$$dr(X) = 4(1-n)\,\psi\,\eta(X), \tag{5.9}$$

where $\psi = tr.\phi$. From (5.9), it follows that

$$dr(X) = 0 \text{ if and only if } \psi = 0.$$

Thus we have the following:

Theorem 5.2. *Let* $(M^n, g)(n > 3)$ *be an LP-Sasakian manifold satisfying the condition* $S(X, \xi) \cdot R = 0$. *Then the scalar curvature of the manifold is constant if and only if the vector field* ξ *is harmonic.*

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