BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **27** (2004) 27–33

A Characterization of Semi Bound Graphs

[1]HIROSHI ERA, [2]KENJIRO OGAWA AND [2,3]MORIMASA TSUCHIYA
[1]Faculty of Information and Communication, Bunkyo University, Chigasaki 253-0007, Japan
[2]Department of Mathematical Sciences, Tokai University, Hiratsuka 259-1292, Japan
[3]Department of Mathematics, MIT, Cambridge, MA02139-4307, USA
e-mail: tsuchiya@ss.u-tokai.ac.jp

Abstract. In this paper we deal with semi bound graphs. For a poset *P*, a graph *G* is a semi bound graph of *P* if $V(G) = V(P)$ and $uv \in E(G)$ if and only if there exists a common upper bound of *u* and *v* or a common lower bound of *u* and *v* in *P*. We characterize semi bound graphs using properties on complete bipartite graphs as induced subgraphs. We also obtain characterizations of triangle-free semi bound graphs and k_4- free semi bound graphs.

2000 Mathematics Subject Classification: 05C62.

1. Introduction

In this paper, we consider finite undirected simple graphs. For a vertex *v* in a graph *G*, the *neighborhood* of *v* is the set of vertices which are adjacent to *v*, and denoted by $N_G(v)$. Cohen [1] introduced graphs associated with food web models, for example, *competition graphs* of a digraph *D*. In the case of transitive digraphs, that is, posets, McMorris and Zaslavsky [4] introduced a concept of upper bound graphs and gave a characterization. Also McMorris, Zaslavsky [4] and Diny [2] deal with double bound graphs.

In [3] we deal with semi bound graphs. For a poset *P*, the *semi bound graph (SB-graph)* of a poset *P* is a graph *G*, where $V(G) = V(P)$ and for distinct vertices *u* and *v* in *G*, $uv \in E(G)$ if and only if there exists a common upper bound of *u* and *v* in *P* or a common lower bound of *u* and *v* in *P*. In [3] we obtain some characterizations of semi bound graphs, using properties of upper bound graphs and double bound graphs. In this paper we consider SB-graphs, using properties on complete bipartite graphs as induced subgraphs.

2. Semi bound graphs

For a poset *P*, $Max\ (P)$ is the set of all maximal elements of *P* and $Min\ (P)$ is the set of all minimal elements of *P*. For $M \subseteq Max(P)$ and $N \subseteq Min(P)$, $H(M, N)$ is the Hasse diagram of the subposet of *P* induced by $M \cup N$.

Lemma 1. *Let P be a poset and G be the semi bound graph of P. For* $\emptyset \neq M \subseteq Max(P)$ *and* $\emptyset \neq N \subseteq Min(P)$, *the induced subgraph* $\langle M \cup N \rangle_{V(G)}$ *is a complete subgraph if and only if the Hasse diagram of the subposet induced by* $M \cup N$ *is a complete bipartite graph.*

Proof. If $H(M, N)$ is a complete bipartite graph, then every element of *N* is a common lower bound of *u* and *v*, for any *u*, $v \in M$, and every element of *M* is a common upper bound of *x* and *y*, for any *x*, $y \in N$. Thus the induced subgraph $\langle M \cup N \rangle_{V(G)}$ is a complete subgraph.

If $H(M, N)$ is not a complete bipartite graph, then there exist $u \in M$ and $x \in N$ such that *u* is not comparable with *x*. Since *u* and *x* have no common upper bound nor common lower bound, $ux \notin E(\langle M \cup N \rangle_{V(G)})$. Thus $\langle M \cup N \rangle_{V(G)}$ is not a complete subgraph.

For a poset *P* and $v \in Max(P)$, let $I_P[v] = \{u \in Min(P);\ u \leq_P v\} \cup \{v\}$. Likewise, for $v \in Min(P), I_P[v] = \{u \in Max(P)\ ;\ v \leq_P u\} \cup \{v\}$. Then the Hasse diagram of $I_P[v]$ is isomorphic to a complete bipartite graph $K_{1,n}$, and $\langle I_P[v] \rangle_{V(G)}$ is a complete graph K_{n+1} by Lemma 1, where $|I_P[v]| = n + 1$. Let $T_n (n \geq 1)$ be the graph with $V(T_n) = \{v_0, v_1, \cdots, v_{n+1}\}$ and $E(T_n) = \{v_i v_{i+1};\ i = 0, 1, \cdots, n\} \cup \{v_i v_{i+2}\ ;\ i = 0, 1, \cdots, n-1\}$. Then T_1 is K_3 and T_2 is $K_4 - e$.

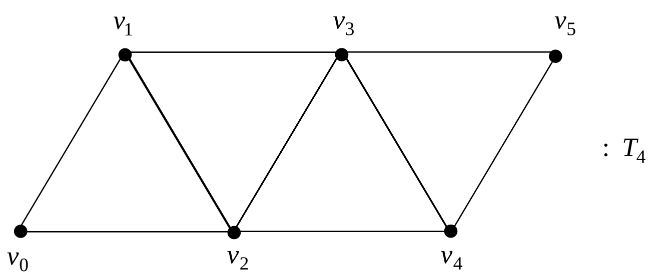

Figure 1. The graph T_4

Let Z_0 be the two elements total ordered set and for $n \geq 1$ let Z_n be the poset with $V(Z_n) = \{v_0, v_1, \cdots, v_{n+1}\}$, where $v_0 \leq_{Z_n} v_1$, $v_{2i} \leq_{Z_n} v_{2i-1}$ and $v_{2i} \leq_{Z_n} v_{2i+1}$ for $i = 1, 2, \cdots, [\frac{n}{2}]$. For a poset *P*, P^d is the dual poset of *P*. By Lemma 1, we have the following result.

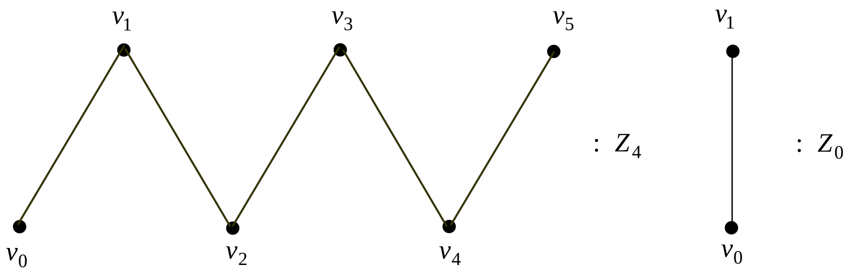

Figure 2. The posets Z_4 and Z_0

Proposition 2. *Let G be a connected semi bound graph.*

(1) *G is triangle-free if and only if G is* K_2,

(2) *G is* K_4*- free if and only if G is* $T_n (n \geq 1)$ *or* K_2.

Proof. (1) K_2 is a connected triangle-free semi bound graph. We assume that *G* is a triangle-free and *P* is a poset with $SB(P) \cong G$. Since *G* has no K_3, for $v \in Max(P)$ (or $v \in Min(P)$), $I_P[v] = Z_0$. If there exists $w \in V(G) - (Max(P) \cup Min(P))$ such that $w \leq_P v$ for some $v \in Max(P)$, then $I_P[v] \cup \{w\}$ corresponds to K_3 in *G*, which is a contradiction. So $V(G) - (Max(P) \cup Min(P)) = \emptyset$, $P = Z_0$ and $G \cong K_2$.
(2) Since $SB(Z_n) \cong T_n$, T_n is a connected K_4- free semi bound graph. Let *G* be a connected K_4- free graph and *P* be a poset with $SB(P) \cong G$. Since *G* is K_4- free, $I_P[v]$ is Z_0 or Z_1 (or Z_1^d) for $v \in Max(P)$ (or $v \in Min(P)$). If $V(G) - (Max(P) \cup Min(P)) = \emptyset$, $P \cong Z_n$ (or $\cong Z_n^d$) $(n \geq 0)$ and $G \cong T_n$ or K_2. Next we consider the case $V(G) - (Max(P) \cup Min(P)) \neq \emptyset$. For $v \in Max(P)$ with $I_P[v] = Z_1$, if there exists $w \in V(G) - (Max(P) \cup Min(P))$ with $w \leq_P v$, then $I_P[v] \cup \{w\} \cong K_4$, which is a contradiction. The proof is similar for $v \in Min(P)$ with $I_P[v] = Z_1^d$. In the case $I_P[v] \cong Z_0$ for $v \in Max(P)$, since *G* is K_4- free, there exists exactly one element $w \in V(G) - (Max(P) \cup Min(P))$ with $w \leq_P v$, $I_P[v] \cup \{w\} = P$ and *G* is K_3.

A *clique* in the graph G is the vertex set of a maximal complete subgraph, and a family $\mathcal{C}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$ there exists $C \in \mathcal{C}$ such that $u, v \in C$. Using Lemma 1, we also have the next result.

Theorem 3. *Let G be a connected graph. Then G is a semi bound graph with* $p \geq 2$ *vertices if and only if there exist disjoint independent vertex subsets M and N satisfying the followings:*

(1) *There exists a family of complete subgraphs* $\mathcal{C}(M \cup N) = \{C_1, \cdots, C_s\}$ *of the induced subgraph* $\langle M \cup N \rangle_{V(G)}$ *such that*

(a) $\mathcal{C}(M \cup N)$ *edge covers* $\langle M \cup N \rangle_{V(G)}$,

(b) For each $i = 1, \cdots, s$ $C_i^M = C_i \cap M$ *and* $C_i^N = C_i \cap N$ *satisfy the following conditions:*

(i) $C_i^M \neq \emptyset$ *and* $C_i^N \neq \emptyset$,

(ii) if $C_i^M \cap C_j^M \neq \emptyset$, *then there exists a* $C_k \in \mathcal{C}(M \cup N)$ *such that* $C_k^M = C_i^M \cap C_j^M$ *and* $C_k^N \supseteq C_i^N \cup C_j^N$, *and*

(iii) if $C_i^N \cap C_j^N \neq \emptyset$, *then there exists a* $C_k \in \mathcal{C}(M \cup N)$ *such that* $C_k^M \supseteq C_i^M \cup C_j^M$ *and* $C_k^N = C_i^N \cap C_j^N$.

(2) *For each vertex* $v \in V(G) - (M \cup N)$, *there exist vertex subsets* M_v *and* N_v *satisfies the following conditions:*

(a) $\emptyset \neq M_v \subseteq M \cap N_G(v)$ *and* $\emptyset \neq N_v \subseteq N \cap N_G(v)$,

(b) for $u \in M$, $uv \in E(G)$ *if and only if there exists* $x \in N_v$ *such that* $ux \in E(G)$,

(c) for $u \in N$, $uv \in E(G)$ *if and only if there exists* $x \in M_v$ *such that* $ux \in E(G)$,

(d) for $w (\neq v) \in V(G) - (M \cup N)$, $uw \in E(G)$ *if and only if* $M_v \cap M_w \neq \emptyset$ *or* $N_v \cap N_w \neq \emptyset$,

(e) there exists $C_i \in \mathcal{C}(M \cup N)$ *such that* $M_v \cup N_v \subseteq C_i$.

Proof. Let G be the connected semi bound graph of a poset P, $M = Max(P)$, $N = Min(P)$ and $\mathcal{C}(M \cup N) = \{C_i; C_i$ is a complete bipartite subgraph of the Hasse diagram of $M \cup N\}$. By Lemma 1, each $C_i \in \mathcal{C}(M \cup N)$ is a complete subgraph of $\langle M \cup N \rangle_{V(G)}$. Since G is connected, $|C_i| \geq 2$ for all i. For $v \in V(P), M_v = \{u \in M; v \leq_P u\}$ and $N_v = \{u \in N; u \leq_P v\}$.

First we consider condition (1)-(a). For $xy \in E(\langle M \cup N \rangle_{V(G)})$, we have four cases as follows:

(1) $x, y \in M$,
(2) $x, y \in N$,
(3) $x \in N, y \in M$,
(4) $x \in N, y \in N$.

From the definition of *M* and *N*, in every case there exists a common upper bound *w* of *x* and *y* or a common lower bound *w* of *x* and *y*. Then $\langle \{x, y, w\} \rangle_P$ form a complete bipartite graph as a Hasse diagram and there exists $C_i \in \mathcal{C}(M \cup N)$ such that *x*, $y \in C_i$. So $\mathcal{C}(M \cup N)$ edge covers $\langle M \cup N \rangle_{V(G)}$.

Next we consider condition (1)–(b). Since $|C_i| \geq 2$ for all *i*, $C_i^M \neq \emptyset$ and $C_i^N \neq \emptyset$. When $C_i^M \cap C_j^M \neq \emptyset$, we define $C_k^M = C_i^M \cap C_j^M$, $C_k^N = \cap_{x \in C_k^M} N_x$ and $C_k = \langle C_k^M \cup C_k^N \rangle_{V(G)}$. Then $\langle C_k^M \cup C_k^N \rangle_P$ is a maximal complete bipartite graph as a Hasse diagram and C_k is a complete subgraph in $\mathcal{C}(M \cup N)$. Since $\langle C_i^M \cup C_i^N \rangle_P$ and $\langle C_j^M \cup C_j^N \rangle_P$ are complete bipartite graphs as a Hasse diagram, $C_k^N \supseteq C_i^N \cup C_j^N$. Likewise for the case, $C_i^N \cap C_j^N \neq \emptyset$, there exists a $C_k \in \mathcal{C}(M \cup N)$ such that $C_k^M \supseteq C_i^M \cup C_j^M$ and $C_k^N = C_i^N \cap C_j^N$.

We consider condition (2). If $u \in M_v$, then $uv \in E(G)$, and if $u \in N_v$, then $uv \in E(G)$, that is, $\emptyset \neq M_v \subseteq M \cap N_G(v)$ and $\emptyset \neq N_v \subseteq N \cap N_G(v)$. For $u \in M$ and $uv \in E(G)$, there exists a common upper bound of *u* and *v* or a common lower bound of *u* and *v*. Since $u \in M$ and $v \notin M \cup N$, there exists $x \in N_v$ such that $ux \in E(G)$. If there exists $x \in N_v$ such that $ux \in E(G)$, there exists $u \in M$ such that $uv \in E(G)$, since $x \in N$ and $v \notin M \cup N$. Similarly, for $u \in N$, $uv \in E(G)$ if and only if there exists $x \in M_v$ such that $ux \in E(G)$.

Let *w* be a distinct vertex from *v* in $V(G) - (M \cup N)$, then $uw \in E(G)$ if and only if there exists a common upper bound of *v* and *w* or a common lower bound of *v* and *w*. This means that $M_v \cap M_w \neq \emptyset$ or $N_v \cap N_w \neq \emptyset$.

Since $\langle M_v \cup N_v \rangle_P$ is a complete bipartite graph as a Hasse diagram, there exists $C_i \in \mathcal{C}(M \cup N)$ such that $M_v \cup N_v \subseteq C_i$. Therefore $\mathcal{C}(M \cup N)$ satisfies the necessary conditions.

Conversely we assume that *G* is a graph with two disjoint independent subset *M*, *N* and a family of complete subgraphs $\mathcal{C}(M \cup N) = \{C_1, \cdots, C_s\}$ satisfying the necessary conditions. Then we obtain a poset *P* as follows:

(1) $V(P) = V(G)$,

(2) for all $x \in V(G), x \leq_P x$,

(3) for each pair $m \in M$ and $n \in N$, $n \leq_P m$ if $mn \in E(G)$, and

(4) for each $v \in V(G) - (M \cup N)$, $v \leq_P m$ for all $m \in M_v$ and $n \leq_P v$ for all $nm \in N_v$.

If $x \leq_P y$ and $y \leq_P z, x \in N_y$ and $z \in M_y$. By condition (2)–(e), there exists a complete subgraph $C_i \in \mathcal{C}(M \cup N)$ such that $x, z \in C_i$. Then $xz \in E(G)$ and $x \leq_P z$. Thus P is a poset on $V(G)$.

Next we show that the semi bound graph of P is G. For any pair of $x, y \in V(G)$, we have six cases as follows:

(1) $x, y \in Max(P) = M$,

(2) $x, y \in Min(P) = N$,

(3) $x \in Max(P) = M$ and $y \in Min(P) = N$,

(4) $x \in Max(P) = M$ and $y \in V(G) - (M \cup N)$,

(5) $x \in V(G) - (M \cup N)$ and $y \in Min(P) = N$, and

(6) $x, y \in V(G) - (M \cup N)$.

Case 1. $x, y \in Max(P) = M$.

If $xy \in E(G)$, there exists a clique C_i such that $x, y \in C_i$ by condition (1). Thus for $v \in C_i \cap N$, $v \leq_P x, v \leq_P y$ and $xy \in E(SB(P))$. If $xy \in E(SB(P))$, there exists a $w \in N$ such that $w \leq_P x$ and $w \leq_P y$. Thus $xw, yw \in E(G)$. By condition (1)–(b)–(iii), there exists $C \in \mathcal{C}(M \cup N)$ such that $x, y, w \in C$. Therefore $xy \in E(G)$.

Case 2. $x, y \in Min(P) = N$.

Similar to Case 1.

Case 3. $x \in Max(P) = M$ and $y \in Min(P) = N$.

By Lemma 1 and the definition of the poset P, $xy \in E(G)$ is equivalent to $xy \in E(SB(P))$.

Case 4. $x \in Max(P) = M$ and $y \in V(G) - (M \cup N)$.

If $xy \in E(G)$, then by condition (2)–(b) there exists $v \in N_y$ and $y \leq_P x$, and $xy \in E(SB(P))$. If $xy \in E(SB(P))$, there exists a $w \in N$ such that $w \leq_P x$ and $w \leq_P y$. Thus $w \in N_y$ and $wx \in E(G)$. Again by condition (2)–(b), $xy \in E(G)$.

Case 5. $x \in V(G) - (M \cup N)$ and $y \in Min(P) = N$.

Using a way similar to that for Case (4) and condition (2)–(c), we can show that $xy \in E(G)$ is equivalent to $xy \in E(SB(P))$.

Case 6. $x, y \in V(G) - (M \cup N)$.

If $xy \in E(G)$, then by condition (2)–(d) $M_x \cap M_y \neq \emptyset$ or $N_x \cap N_y \neq \emptyset$. Thus $xy \in E(SB(P))$. If $xy \in E(SB(P))$, there exists a $w \in M$ such that $x, y \leq_P w$ or $w \in N$ such that $w \leq_P x, y$. In the former case $w \in M_y$ and $w \in M_x$. Thus $M_x \cap M_y \neq \emptyset$. By condition (2)–(d), $xy \in E(G)$. In the latter case, $xy \in E(G)$ similarly.

References

1. J.E. Cohen, *Interval graphs and food webs: A finding and a problem*, RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.
2. D. Diny, The double bound graph of partially ordered set, *Journal of Combinatorics, Information & System Sciences* **10** (1985), 52–56.
3. H. Era, K.Ogawa and M. Tsuchiya, A note on semi bound graphs, *Congressus Numerantium* **145** (2001), 129–135.
4. F.R. McMorris and T. Zaslavsky, Bound graphs of a partially ordered set, *Journal of Combinatorics, Information & System Sciences* **7** (1982),134–138.