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Superposition Operators on $W_0(\phi)$

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Abstract. For any linear operator on a Banach space, the continuity and boundedness are equivalent. It fails for a nonlinear operator, in particular, for a superposition operator. In this paper, we present sufficient and necessary conditions in terms of an inequality for the continuity of a superposition operator on the function space $W_0(\phi)$.

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1. Introduction

Let N and R denote the set of all positive integers and the real numbers respectively. If $g(k,t) : N \times R \to R$, then a superposition operator P_g from a sequence space of real numbers X into another one is defined by:

$$P_g(\{x_k\}) = \{g(k, x_k)\},$$

for every $\{x_k\} \in X$. Under the assumption that $g(k,\cdot)$ is continuous on R for every k, Chew and Lee [1] have characterized $P_g : l_p \to l_1, 1 \le p < \infty$, and $P_g : c_0 \to l_1$. Meanwhile Paredes [2] has characterized $P_g : w_o(\Phi) \to l_1$ where $w_o(\Phi)$ is the sequence version of the function space $W_0(\phi)$ defined below. The sufficient and necessary condition for the continuity of P_g on the sequence spaces have also been given in [1] and [2]. The purpose of this paper is to generalize these results to a function space.

Let $\boldsymbol{m}[a,\infty)$ denote the collection of all measurable real-valued functions on $[a,\infty)$. For each $x \in [a,\infty)$ and $f \in \boldsymbol{m}[a,\infty)$, the function f_x is defined as follows :

$$f_x(t) = \begin{cases} f(t) & \text{whenever } a \le t \le x, \\ 0 & \text{otherwise} \end{cases}$$

It is clear that for every $x \in [a,\infty)$, we have $f_x \in \boldsymbol{m}[a,\infty)$, whenever $f \in \boldsymbol{m}[a,\infty)$.

In this paper we always assume that $X \subset \boldsymbol{m}[a,\infty)$.

Let X be a linear space of functions on $[a,\infty)$ over real numbers R. A non-negative function $\|\cdot\| : X \to [0,\infty)$ is called an ***F-norm*** if for every $f, g \in X$ we have:

(i) $\| f \| = 0 \Leftrightarrow f = 0$,

(ii) $\| -f \| = \| f \|$,

(iii) $\| f + g \| \le \| f \| + \| g \|$,

(iv) If $\{a_n\} \subset R$ is any sequence which converges to a and $\{f^{(n)}\}$ is any sequence in X such that $\| f^{(n)} - f \| \to 0$, for some $f \in X$, then $\| a_n f^{(n)} - af \| \to 0$.

As usual, $f = 0$ means $f(x) = 0$ almost everywhere in $[a,\infty)$. Furthermore, $(X, \|\cdot\|)$ i.e. a linear space equiped with an *F-norm* is called an ***F-normed Space***. It is easy to prove that a function $d : X \times X \to [0,\infty)$ given by $d(f,g) = \| f - g \|$ is a metric on X. If an ***F-normed Space*** X is complete then it is called a ***Frèchet Space*** or an ***F-Space*** in short.

A space X is called an ***FK-Space*** if it is an F-space and the canonical mapping $p_x : X \to R$,

$$p_x(f) = f(x), \qquad f \in X$$

is continuous for every $x \in [a,\infty)$, in the sense that if $\| f^{(n)} - f \| \to 0$ then $f^{(n)}(x) \to f(x)$ as $n \to \infty$.

An F-space X is called an ***AK-Space*** if X contains χ_E and f_x for any set $E \subset [a,\infty)$ of finite measure, $f \in X$ and $x \in [a,\infty)$, and $\lim_{x\to\infty} \| f_x - f \| = 0$.

An F-normed space X is said to be ***solid*** if for every $f \in X$ and $g \in m[a,\infty)$ such that if $|g(x)| \le |f(x)|$ for almost all $x \in [a,\infty)$, then $g \in X$.

A real-valued function $\phi : R \to [0,\infty)$ is called a ***ϕ-function*** if it is continuous, even, increasing on $[0,\infty)$, and $\phi(t) \to \infty$, as $t \to \infty$. We shall always assume that the ϕ-function ϕ satisfies the Δ_2- ***condition***, i.e. there exists a real number $M > 0$ such that: $\phi(2t) \le M\phi(t)$ for every $t \ge 0$ (see [5]).

Further, fix a real number $a > 0$ and we define the space:

$$W_0(\phi) = \left\{ f \in \boldsymbol{m}\,[a,\infty)\,;\ \lim_{s\to\infty} \rho_s(f) = 0 \right\},$$

where ρ_s is given by :

$$\rho_s(f) = \frac{1}{s}\int_a^s \phi\big(f(x)\big)dx\,, \quad s \in [a,\infty).$$

We can show that ρ_S is a modular on $W_0(\phi)$, i.e. it is even, increasing on the set $\{f \in \boldsymbol{m}[a,\infty);\ f \geq 0 \text{ a.e on } [a,\infty)\}$ and satisfies the following conditions:

(i) $\rho_s(f) = 0 \Leftrightarrow f = 0$ a.e on $[a,\infty)$,

(ii) $\rho_s(\max\{f,g\}) \leq \rho_s(f) + \rho_s(g)$, for every $f, g \in W_0(\phi)$.

If we define the function ρ on the space by:

$$\rho(f) = \sup\{\rho_s(f),\ \ s \in [a,\infty)\},$$

we can also prove that ρ is a modular on the space. Supama and Soeparna [3] observed that $W_0(\phi)$ is a solid *FK*- and *AK*-space with respect to:

$$\| f \| = \inf\left\{\varepsilon > 0\ ;\ \rho\left(\frac{f}{\varepsilon}\right) \leq \varepsilon\right\}, \quad f \in W_0(\phi).$$

Let *X* be a function space. A functional $F : X \to R$ is said to be ***continuous*** at $f \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that :

$$\left| F(f) - F(h) \right| < \varepsilon\,,$$

whenever $\| f - h \| < \delta$. Further, *F* is said to be ***continuous*** on *X* if it is continuous at every $f \in X$.

Two functions $f,\ g \in X$ are said to be ***orthogonal*** or ***disjoint***, written by $f \perp g$, if $f(x) \cdot g(x) = 0$ for almost all $x \in [a,\infty)$. The functional $F : X \to R$ is said to be ***orthogonally additive*** if for every $f,\ g \in X$,

$$F(f + g) = F(f) + F(g),$$

whenever $f \perp g$. Then, Supama and Soeparna (see [3], [4]) characterized the following:

Theorem 1.1. *The functional* $F : W_0(\phi) \to R$ *is continuous and orthogonally additive if and only if there exists* $g(x,t) : [a,\infty) \times R \to R$ *satisfying that* $g(\,.\,,t)$ *is measurable for every* $t \in R$, $g(x,0) = 0$ *and* $g(x,\cdot)$ *is continuous on* R *for every* $x \in [a,\infty)$ *such that*:

$$F(f) = \int_a^\infty g(x, f(x))\,dx$$

exists for every $f \in W_0(\phi)$.

2. Some lemmas

The main results of the research are Theorem 3.1. The following lemmas are required for proving those theorems.

Lemma 2.1. *Let* $f \in W_0(\phi)$. *Then for every* $\beta > 0$ *there exists* $\alpha > 0$ *such that* $\rho(f) \le \beta$, *whenever* $\|f\| \le \alpha$.

Proof. Let $f \in W_0(\phi)$ and $\beta > 0$ be given. There exists $n_0 \in N$ such that $\beta \le 2^{n_0}$. So, we have:

$$\begin{aligned}
\rho(f) &= \sup_{s\ge a} \left\{ \frac{1}{s} \int_a^s \phi\big(f(x)\big)dx \right\} \\
&\le \sup_{s\ge a} \left\{ \frac{1}{s} \int_a^s \phi\left(\frac{2^{n_o}}{\beta} f(x)\right)dx \right\} \\
&\le M^{n_o} \sup_{s\ge a} \left\{ \frac{1}{s} \int_a^s \phi\left(\frac{1}{\beta} f(x)\right)dx \right\}, \quad \text{for some } M > 0 \\
&= M^{n_o}\, \rho\left(\frac{f}{\beta}\right)
\end{aligned}$$

If we choose $\alpha > 0$ such that $\alpha < \beta$ and $M^{n_o}\alpha < \beta$ then the assertion follows.

Lemma 2.2. *Let* $f \in W_0(\phi)$. *Then for every real numbers* $\alpha, \gamma > 0$ *there exists* $\beta > 0$ *such that* $\|f\| \le \alpha$, *whenever* $\rho(\gamma f) \le \beta$.

Proof. Let $\alpha, \gamma > 0$, then there exists $n_0 \in N$ such that $\frac{1}{\alpha} \le \gamma 2^{n_o}$. Since ϕ satisfies the Δ_2-condition then for every $s \in [a,\infty)$ we have:

$$\rho_s\left(\frac{f}{\alpha}\right) = \frac{1}{s}\int_a^s \phi\left(\frac{1}{\alpha} f(x)\right)dx$$
$$\le \frac{1}{s}\int_a^s \phi\left(2^{n_o}\gamma f(x)\right)dx$$
$$\le \frac{M^{n_o}}{s}\int_a^s \phi\left(\gamma f(x)\right)dx, \text{ for some } M > 0$$
$$= M^{n_o}\rho_s(\gamma f)$$

It implies $\rho\left(\frac{f}{\alpha}\right) \le M^{n_o}(\gamma f)$. If we choose $\beta \le \dfrac{\alpha}{M^{n_o}}$ then $\rho\left(\frac{f}{\alpha}\right) \le \alpha$, as $\rho(\gamma f) \le \beta$. So, $\| f \| \le \alpha$, whenever $\rho(\gamma f) \le \beta$. This completes the proof.

We remark that the above two lemmas remain valid with $W_0(\phi)$ replaced by a modulared space *X* (see [5]). We presented only the case when $X = W_0(\phi)$ as it is the form required later.

Lemma 2.3. *Let X be an FK- and AK-space. Suppose* $g(x,t) : [a,\infty) \times R \to R$ *satisfies the following conditions:* $g(.,t)$ *is measurable for every* $t \in R$, $g(x,0) = 0$ *and* $g(x,\cdot)$ *is continuous on* R *for every* $x \in [a,\infty)$. *If there exist real numbers* α, $\beta > 0$ *such that* $\frac{1}{s}\int_a^s \phi(f(x))\,dx \le \beta$ *implies* $\int_a^s \left| g(x, f(x)) \right| dx \le \alpha$ for each $f \in X$ and $s \in [a,\infty)$ *then for every* $s \in [a,\infty)$ *there is a non-negative function* $h \in L_1[a,s]$, *depending on s, with* $\int_a^s h(x)\,dx \le \alpha$ *such that for every* $x \in [a,s]$,

$$\left| g(x,t) \right| \le h(x) + 2\alpha\beta^{-1} s^{-1} \phi(t),$$

whenever $\frac{\phi(t)}{s} \le \beta$.

Proof. Let $s \in [a,\infty)$ and $t \in R$ be given. We define the function *k* on $[a,s] \times R$ ⊊and the function *h* on $[a,s]$ as follows :⊊⊊

$$k(x,t) = \begin{cases} |\, g(x,t) \,| - 2\alpha\beta^{-1} s^{-1} \phi(t) & \text{when } |\, g(x,t) \,| \ge 2\alpha\beta^{-1} s^{-1} \phi(t), \\ 0 & \text{otherwise} \end{cases}$$

$$h(x) = \sup\left\{ k(x,t);\ \frac{\phi(t)}{s} \le \beta \right\} = k(x,u(x)).$$

Note that both functions k and h depend on s. Since ϕ is continuous and has an inverse, $h(x)$ above is uniquely determined. It is clear that $h(x) \geq 0$ for every $x \in [a,s]$. Since, $g(x,\cdot)$ and ϕ are both continuous on R then $k(x,\cdot)$ is continuous on R for every $x \in [a,s]$. Further, $\left\{t\,;\, \frac{\phi(t)}{s} \leq \beta\right\}$ is closed and bounded, in view of the continuity of ϕ . Hence, we have $h(x)$ is finite for $x \in [a,s]$. Next we prove $h \in L_1[a,s]$.

For each $s \in [a,\infty)$, we can decompose as follows:

$$\int_a^s \phi(u(x))dx = \int_{A_1} \phi(u(x))dx + \int_{A_2} \phi(u(x))dx + \cdots + \int_{A_n} \phi(u(x))dx$$

where $A_1, A_2, \cdots, A_n$ are measurable subsets of $[a,s]$ with $\bigcup_{i=1}^{n} A_i = [a,s]$, the Lebesgue measure $\mu(A_i \cap A_j) = 0$ for every $i \neq j$, $i, j = 1,2,\cdots,n$, and $\frac{\beta}{2} \leq \frac{1}{s}\int_{A_i} \phi(u(x))\,dx \leq \beta$, $i = 1,2,3,\cdots,n-1$ and $0 \leq \frac{1}{s}\int_{A_n} \phi(u(x))\,dx \leq \beta$. So, we have:

$$\begin{aligned}
\int_a^s h(x)\,dx &= \int_a^s k(x,u(x))\,dx \\
&\leq \int_a^s |\, g(x,u(x))\,|\,dx - 2\alpha\beta^{-1}s^{-1}\int_a^s \phi(u(x))\,dx \\
&= \sum_{i=1}^{n}\int_{A_i} |\, g(x,u(x))\,|\,dx - 2\alpha\beta^{-1}s^{-1} \sum_{i=1}^{n}\int_{A_1} \phi(u(x))\,dx \\
&\leq n\alpha - 2\alpha\beta^{-1}s^{-1}(n-1)\left(\beta/2\right) = \alpha\,.
\end{aligned}$$

Thus, $h \in L_1[a,s]$ and $\int_a^s h(x)\,dx \leq \alpha$.

From the definitions of h and $k(x,t)$, we have $k(x,t) \leq h(x)$ for every $x \in [a,s]$, whenever $\frac{\phi(t)}{s} \leq \beta$. So, by the definition of $k(x,t)$ we have:

$$|\, g(x,t)\,| \leq h(x) + 2\alpha\beta^{-1}s^{-1}\,\phi(t),$$

whenever $\frac{\phi(t)}{s} \leq \beta$. This completes the proof.

3. Main result

Let X be a function space and $g(x,t):[a,\infty)\times R\to R$. We define a superposition operator P_g on X as follows:

$$P_g(f)(x) = g(x, f(x)),$$

for every $f\in X$ and $x\in[a,\infty)$. It is clear that the conditions of X required in Lemma 2.3 are all satisfied by the space $W_0(\phi)$ (see [3]). Hence, we can prove the following theorem.

Theorem 3.1. *Let $g(x,t):[a,\infty)\times R\to R$ be given such that $g(.,t)$ is measurable for every t, $g(x,0)=0$ and $g(x,\cdot)$ is continuous on R for every $x\in[a,\infty)$. The operator $P_g: W_0(\phi)\to L_1[a,\infty)$ if and only if there exist $\alpha,\beta>0$ and for every $s\in[a,\infty)$ there exists non-negative function $h\in L_1[a,s]$, depending on s, with $\int_a^s h(x)\,dx\le\alpha$ such that for every $x\in[a,s]$,*

$$|g(x,t)|\le h(x)+2\alpha\beta^{-1}s^{-1}\phi(t),$$

whenever $\frac{\phi(t)}{s}\le\beta$.

Proof.
(⇐): Let $f\in W_o(\phi)$, then there exists a real number $M>a$ such that for every $s\ge M$:

$$\frac{1}{s}\int_a^s \phi(f(x))\,dx < \beta\,.$$

It follows that:

$$\int_a^s |g(x, f(x))|\,dx \le \int_a^s h(x)\,dx + 2\alpha\beta^{-1}s^{-1}\int_a^s \phi(f(x))\,dx \le \alpha + 2\alpha\beta^{-1}\beta = 3\alpha,$$

for every $s\ge M$. Hence, $\int_a^\infty |g(x, f(x))|$ exist.

(⇒): By Theorem 1.1, the functional $F: W_o(\phi)\to R$ given by:

$$F(f)=\int_a^\infty |g(x, f(x))|\,dx\,,\quad f\in W_o(\phi)$$

is continuous and orthogonally additive. So, there exists $\eta > 0$ such that for each $f \in W_o(\phi)$,

$$\int_a^\infty |g(x, f(x))| \, dx < 1,$$

whenever $\| f \| < \eta$. Also, by Lemma 2.2 there exists $\beta > 0$ such that $\| f \| < \eta$ as $\rho(f) \le \beta$. Thus, for $\rho(f) \le \beta$ we have:

$$\int_a^\infty |g(x, f(x))| \, dx < 1.$$

It means that for every $s \in [a, \infty)$,

$$\int_a^s |g(x, f(x))| \, dx < 1,$$

whenever

$$\frac{1}{s} \int_a^s \phi(f(x)) \, dx \le \beta .$$

By choosing

$$\alpha = \sup \left\{ \int_a^s |g(x, f(x))| \, dx \, ; \, \tfrac{1}{s} \int_a^s \phi(f(x)) dx \le \beta \right\}$$

then we have:

$$\int_a^s |g(x, f(x))| dx \le \alpha,$$

whenever $\frac{1}{s} \int_a^s \phi(f(x)) \, dx \le \beta$. By Lemma 2.3 there exists a non-negatif function $h \in L_1[a, s]$, depending on s, with $\int_a^s h(x) \, dx \le \alpha$ such that for every $x \in [a, s]$,

$$|g(x, t)| \le h(x) + 2\alpha\beta^{-1} s^{-1} \phi(t),$$

whenever $\frac{\phi(t)}{s} \le \beta$. This completes the proof.

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