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The Euclidean Inscribed Polygon

RICHARD L. FRANCIS
Southeast Missouri State University, Cape Girardeau, MO 63701
e-mail: c714scm@semo.edu

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1. Introduction

Euclid's work reflected in many ways his Pythagorean predecessors and was to prove of lasting impact. Yet these early geometric developments did not utilize the tools of sexagesimal angle measure. Incorporation of the partitioning of the circle into 360 equal parts, largely the work of Hipparchus in his development of formalized trigonometry, was not introduced into Greece until 140 B.C. or so (some one to two centuries following Euclid). Although the sexagesimal system of degrees, minutes, and seconds was slow in being absorbed into early Greek mathematics, the language of this system now prevails in discussing the geometry of Euclid. Such a method of expression becomes evident in pursuit of key books of the *Elements* and stands out clearly in modern-day commentaries on Book 4 [1]. This book of sixteen propositions concerns regular polygon constructions and figures circumscribed about or inscribed in a circle, and foreshadows many centuries of mathematical advancement.

Regular polygon constructibility standards are today well known and extend far beyond the content of Book 4. These enhancements reflect Gauss' work in his 1801 *Disquisitiones Arithmeticae* and the writings of others in the early nineteenth century (reference 6). Present insights stand in contrast to the construction limitations of the long-ago Greeks. Ancient geometers knew how to construct not only the regular triangle, quadrilateral, and hexagon, but the pentagon as well (a result growing out of the known constructibility of the golden mean). The regular pentadecagon (15-sided polygon) was also known to be constructible, but the regular heptadecagon (17-sided polygon) was then a mystery. Constructibility of the latter proved a remarkable discovery by a young Gauss in 1796.[1]

Note in Figure 1 the ancient method of constructing what is today known as the 24-degree angle. As this is the central angle of a regular pentadecagon, the long-ago

[1] This was a mathematically significant decade, a time period in which the Law of Quadratic Reciprocity was verified as well as a first proof of the Fundamental Theorem of Algebra.

Euclidean construction is complete. Repeated bisecting of the 24-degree angle yields a 3-degree result (the smallest constructible angle of positive integral degree). Significantly, an angle of integral degree measure is constructible if and only if its measure is a multiple of 3 [2].

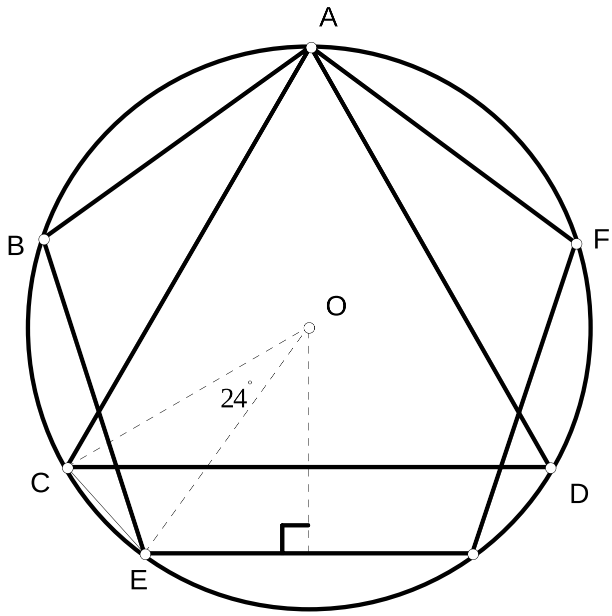

Figure 1. The regular triangular-pentagonal method which forms the side $\overline{CE}$ of a regular pentadecagon

2. An unusual pentadecagon

Today, there are only 31 regular polygons of an odd number of sides which are known to be constructible. These polygons extend from the 3-sided figure to the one of (3)(5)(17)(257)(65537) or 4294967295 sides [5]. The limitation is based on present day uncertainty concerning the number of Fermat primes (i.e., primes of the form $2^n + 1$). However, the *regular pentadecagon* was the regular polygon of the greatest odd number of sides in the constructible category known to the early Greeks. Because of the ancient limitation, focus will be placed on a select pentadecagon easily constructible by the Euclidean tools, a figure possessing quite an assortment of interesting properties. It should be noted that some of the inscribed figures of 15 sides admit a trivial construction (such as repeatedly bisecting the circular arc and consecutively joining any 15 of its points (vertices).

This interesting sexagesimal variation on constructible pentadecagon types is that of the *Euclidean inscribed polygon*. It is defined simply as a constructible polygon inscribed in a circle whose central angles form an integral arithmetic sequence with a non-zero difference.

If the angle presently identified as 3 degrees is itself considered a basic unit, then the central angle measures become respectively $1, 2, 3, \ldots, 15$ (a listing overtly agreeing with counting the sides). In recent literature, this angular unit is identified as the "tride," a word contraction of the "tri-degree" angle. Construction restrictions are designated by use of the word "Euclidean," a standard reference to the allowable curves of construction (i., e., the line and the circle).

Euclid's inscribed polygon contains an assortment of conditions which raise fundamental questions. First, does Euclid's inscribed pentadecagon exist, and if so, what other properties does it have? These extend from elementary questions to the more difficult concerns of generalization.

Solution 2.1.

1. Recall that the first *n* terms of an arithmetic series can be summed by the formula $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$. In this case, *a* is the first term and *d* is the common difference.
2. As $n = 15$, then $S_{15} = \frac{15}{2}\left[2a + 14d\right] = 360$ or $a + 7d = 24$.
3. The only positive integral solutions of this Diophantine equation are $a = 3$, $d = 3$, and $a = 10$, $d = 2$ and $a = 17$, $d = 1$.
4. Only the first $(a = 3^\circ)$ is constructible. The polygon thus exists and is here shown (Figure 2) inscribed in the unit circle.

It should be noted in passing that the angle sum is $(15-2)(180^\circ)$ or 2340°. Moreover, the perimeter, if the typical side is X_i, is given by $\sum_{i=1}^{15} \sqrt{2[1 - \cos 3i^\circ]}$ (law of cosines) whereas the area is $\frac{1}{2}\sum_{i=1}^{15} \sin 3i^\circ$.

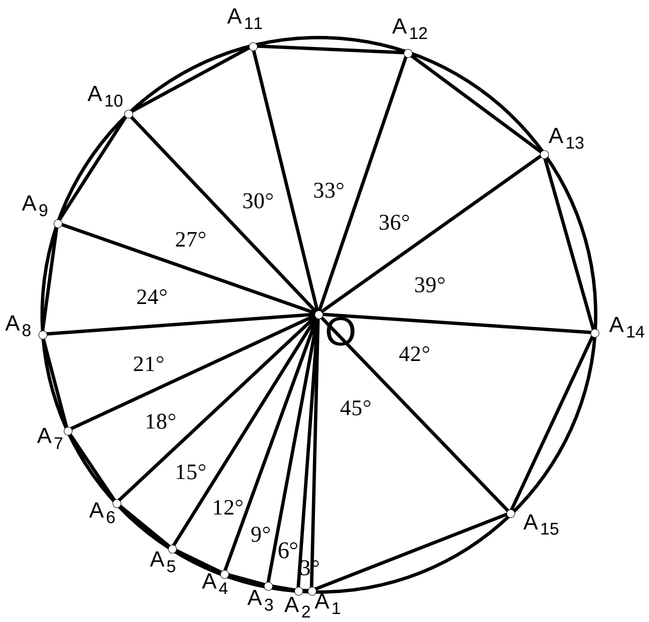

Figure 2. Euclidean inscribed Pentadecagon (whose central angles form an integral arithmetic sequence with a non-zero difference)

This polygon, like all other polygons, can be squared by direct use of the unmarked straightedge and compass. Excluding the least interior angle, the remaining interior angles, in numerical order, form an arithmetic sequence. Sequence terms are not necessarily integers.

It is easy to show that Euclid's inscribed polygon exists for both the triangle and the pentagon. For example, an inscribed triangle stems from the central angles 117°, 120°, 123° (a constructible arithmetic sequence whose sum is 360°). For the pentagon, consider the central angles given by $66^\circ, 69^\circ, 72^\circ, 75^\circ, 78^\circ$. This observation leads to a key result involving such polygons of a prime number of sides.

Theorem 2.1. *Euclid's inscribed n-gon does not exist for any prime n greater than 5.*

Solution 2.2.

Suppose n is a prime and $n > 5$. Then $S_n = \frac{n}{2}[2a + (n-1)d] = 360$ or $2an + n(n-1)d = 720$. As n is a divisor of the left member but not a divisor of 720 (whose only odd prime divisors are 3 and 5), the Diophantine equation in a and d is not solvable. Hence, no such n-gon exists. Accordingly, no 17-sided counterpart to Gauss' famous regular heptadecagon can be found [3].

A similar analysis leads to a more generalized result as follows:

Theorem 2.2. *If n is an odd integer greater than 15, no n-gon, constructible or not, has a Euclidean inscribed n-gon counterpart.*

Rather trivially, it should be noted that the set of Euclidean inscribed polygons is a finite set. Such a result stems immediately from the positive integral restriction in the definition. Because of this, certain minor results are noted, including the following:

Corollary 2.1. *Euclid's inscribed n-gon does exist for some but not all even numbers n where* $2 < n < 15$.

Solution 2.3.

The proof is by counterexample. Suppose $n = 12$. Then $\frac{12}{2}[2a + 11d] = 360$ or $2a + 11d = 60$. This linear Diophantine equation has $a = 19$, $d = 2$ and $a = 8$, $d = 4$ as its only positive integral solutions. Neither a value represents a constructible angle. Hence, the Euclidean inscribed n-gon does not exist for all even numbers n. But it does exist for some. For example, let $n = 6, a = 45$, and $d = 6$. Or, let $n = 4, a = 81$, and $d = 6$ (Figure 3). One sees evidence in this last example that opposite angles of an inscribed quadrilateral are supplementary.

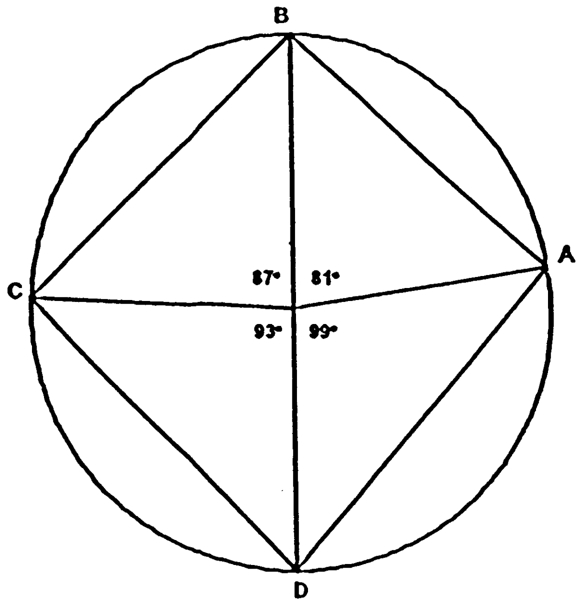

Figure 3. The Euclidean Inscribed 4-Gon ($A = 90^\circ, B = 96^\circ, C = 90^\circ, D = 84^\circ$)

3. Connections

It is evident that infinitely many *n*-gons are constructible for any select $n > 2$. Ancient Greek mathematicians were fully aware of this. However, they were in doubt should it be stipulated that these *n*-gons must also be regular. Significantly, the Euclidean inscribed *n*-gon and the regular *n*-gon have a constructiblity connection, though clearly it is not one of a necessary and sufficient kind. It is expressed concisely in the following:

Theorem 3.1. *If the Euclidean inscribed n-gon exists, then the regular n-gon is constructible (but not conversely).*

Solution 3.1.

As $\frac{n}{2}[2a + (n-1)d] = 360$, the integer n must be a divisor of 720 (which is $2^4 \cdot 3^2 \cdot 5$). The only excluded values in this factorization are multiples of 9 (those values violating Gauss' constructibility standard as the Fermat prime 3 is used as a factor twice). Proceeding by unacceptable cases ($n = 9, 18, 36, 144,$ and 45) and noting that $3 \mid a$, $3 \mid d$, the left member remains a multiple of 3 but the right member is not. Note for $n = 9$ that $\frac{9}{2}[2a + 8d] = 360$ or by simplification that $a + 4d = 40$. As 3 is not a divisor of 40, this case is complete. In like manner, the remaining *n* values can be rejected.

This establishes by the contrapositive that if a regular *n*-gon is not constructible, then the corresponding Euclidean inscribed polygon does not exist.

Sufficiency of the condition is thus noted. Evidently, the converse fails by simple consideration of the known constructibility of the 257-gon and the 65537-gon.[2] Interestingly, the detailed steps of these latter two constructions are attributed respectively to R.J. Richelot in 1832 and Professor O. Hermes around 1894 [3].

4. The generalized Euclidean polygon

Suppose the integral restriction on angle measures and consecutive differences is removed. As the term "Euclidean" in this context pertains to use of the "Euclidean instruments," the constructible features of these angles are preserved. Note that if all the central angles must be constructible, then *d* is necessarily constructible. The formula

$$\frac{n}{2}[2a + (n-1)d] = 360,$$

[2] By the Gaussian constructibility criterion, a regular polygon of an odd number of sides *n* is constructible if and only if *n* is a Fermat prime (of the form $2^r + 1$) or the product of distinct Fermat primes. Only 5 Fermat primes are known today, namely, 3, 5, 17, 257, and 65537.

reduced to $2a + (n-1)d = \frac{720}{n}$,

provides a place of beginning. Without loss of generality, n will be considered an odd number.

As a and d are constructible, it follows that n must be a Fermat product (i.e., product of distinct Fermat primes) in order to guarantee the constructibility of $\frac{720}{n}$ [6]. This result reflects the fact that the set of constructible angles is closed with respect to addition. Since

$$2a < \frac{720}{n}, \text{ then } a < \frac{360}{n}.$$

But $n \geq 3$. Hence, $a < 120$.

To illustrate, suppose $n = 17$ and $a = \frac{3}{16}$ (of a degree). Then $2(\frac{3}{16}) + 16d = \frac{720}{17}$ or $d = \frac{\frac{720}{17} - \frac{3}{8}}{16}$ (clearly a constructible difference as predicted).

Though d is necessarily constructible, it still may be hard to actually perform the construction. Of course, if n is a Fermat prime, then $n-1$ is a power of 2 and the work reduces to repeated bisecting. However, if n is a product of (distinct) Fermat primes, more effort is required. For example, if $n = 15$ (a Fermat product) and $a = \frac{3}{8}$ (a constructible angle), then $d = \frac{27}{8}$ (clearly constructible).

Such a discussion of non-integral angle measures opens the door to number complexities not here pursued. It is an area in which algebraic and transcendental classifications appear and, in some measure, touches on various unsolved problems in mathematics today [4]. Further generalizations provide an exploration of Euclidean inscribed polygons by altering the sequence type but yet preserving constructibility. One may wish, for example, to explore such polygons with restrictions made to the geometric rather than the arithmetic sequence approach as pursued above.

5. Conclusion

Constructibility proves a cornerstone of early demonstrative mathematics. Inclusion of angle measure techniques in the overall development provides an integrated and appealing challenge. Such fundamental notions from the distant past are symbolized impressively by the Euclidean inscribed polygon – and bring into focus the time-scattered topics of Babylonian sexagesimal measure, the basic tools of Greek geometers, and key theorems of modern-day number theory.

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