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On the Value Distribution of $f^2 + af^{(k)}$

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Abstract. In this paper, we improve a result of H.H. Chen by proving that if f is a transcendental integral function, then, for any finite non-zero complex number a and any positive integer k, $f^2 + af^{(k)}$ assumes every finite complex value infinitely often.

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1. Introduction

One of the most important results in the value distribution theory is the following theorem of Hayman.

Theorem A. [3] *If g is a transcendental meromorphic function, then, either g itself assumes every finite complex value infinitely often, or $g^{(k)}$ assumes every finite non-zero value infinitely often for any positive integer k.*

As a consequence of Theorem A, we have

Theorem B. *If f is a transcendental integral function, then $f^2 + af'$ has infinitely many zeros for any finite non-zero complex value a.*

In fact, for an integral function f, $g = 1/f$ has no zeros and the zeros of $g' - 1/a$ are zeros of $f^2 + af'$.

Let f be a meromorphic function in the complex plane. We say that f is a Yosida function if there exists a positive number M such that $f^{\#}(z) \le M$ for all $z \in C$, where

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative.

Ye [7], Chen and Hua [2] independently proved that Theorem B can be generalized by substituting $f^{(k)}$ for f'. In 1996, Chen [1] proved a stronger conclusion for $k = 2$ in the case that f is not a Yosida function.

Theorem C. [1] *Let f be a transcendental integral function. If f is not a Yosida function, then, for any finite non-zero complex number a and any positive integer k, $f^2 + af^{(k)}$ assumes every finite complex value infinitely often.*

It is natural to ask: *What could we say about Theorem C for the case that f is a Yosida function?* In this paper, we shall use a different method but much simpler to prove that Theorem C still holds without the requirement that f is not a Yosida function.

We shall use the standard notations in Nevanlinna theory (cf.[4],[6]).

2. Lemma

To prove our result, we need the following lemma.

Lemma 1. [5] *Let f be a non-constant meromorphic function, and let $\psi = f^n + P[f]$, where $n \geq 2$ and $P[f]$ is a differential polynomial of f with degree $d(P) \leq n-1$. Then we have at least one of the following:*

(a) there exists a small proximity function $c(z)$ (that is, $m(r,c) = S(r,f)$) such that $\psi = \alpha(f + c/n)^n$ and

$$N(r,c) \leq \max\{0, w(P) - n + 1\}\left(\overline{N}(r,1/\psi) + \overline{N}(r,f)\right) + S(r,f),$$

where α is a non-zero constant and $\alpha = 1$ provided that $N(r,f) = S(r,f)$; here $w(P)$ is the weight of P;

(b) $$T(r,f) \leq \left(1 + 2\max\{0, w(P) - n + 1\}\right)\overline{N}(r,1/\psi)$$
$$+ \left(3 + 2\max\{0, w(P) - n + 1\}\right)N(r,f) + S(r,f).$$

3. Main result

Theorem 1. *If f is a transcendental integral function, then, for any finite non-zero complex number a and any positive integer k, $f^2 + af^{(k)}$ assumes every finite complex value infinitely often.*

Proof. We only need to prove that $f^2 + af^{(k)} - b$ has infinitely many zeros for any finite complex number b. Suppose that $f^2 + af^{(k)} - b$ has only finitely many zeros, then

$$N\left(r, 1/\left(f^2 + af^{(k)} - b\right)\right) = O(\log r) = S(r, f).$$

Next we prove that

$$T(r, f) \le (2k+1)\overline{N}\left(r, 1/\left(f^2 + af^{(k)} - b\right)\right) + S(r, f). \quad (1)$$

Put $P[f] = af^{(k)} - b$ and $\psi = f^2 + P[f]$, then $d(P) = 1$ and $w(P) = k+1$. It is easy to see that case (b) of Lemma 1 is what we need. Thus, by Lemma 1, we only need to consider case (a).

Since $N(r, f) = 0$, then $\alpha = 1$. Hence

$$\psi = \left(f + \frac{c}{2}\right)^2, \quad (2)$$

where $c(z)$ is a small proximity function. From (2) and the definition of ψ we get

$$c^2 = 4af^{(k)} - 4cf - 4b \quad (3)$$

From (3), we see that c has no poles. Then

$$T(r, c) = S(r, f).$$

Let $u = f + c/2$, then (2) becomes

$$\psi = u^2. \quad (4)$$

Substituting $f = u - c/2$ into (3) we obtain

$$c^2 - 2ac^{(k)} = 4cu - 4au^{(k)} + 4b. \quad (5)$$

Now we consider two cases.

Case 1. $c^2 - 2ac^{(k)} \equiv 0$. If c is not a constant, then $c = 2ac^{(k)}/c$. Thus

$$T(r, c) = m(r, c) = S(r, c),$$

which is a contradiction. If c is a constant, then $c \equiv 0$ by the hypothesis of Case 1. From this and (2), we deduce that f is a polynomial, a contradiction.

Case 2. $c^2 - 2ac^{(k)} \not\equiv 0$. Set $v(z) = c^2 - 2ac^{(k)}$. From (5) we get

$$\frac{1}{u} = \frac{4c}{v} - \frac{4a}{u}\frac{u^{(k)}}{u} + \frac{4b}{vu},$$

that is,

$$\frac{1}{u} = \frac{4c}{v-4b} - \frac{4a}{v-4b}\frac{u^{(k)}}{u}. \tag{6}$$

Noting that c and v are small functions of f, from (6) we have

$$m\left(r, \frac{1}{u}\right) \le S(r,f) + S(r,u) = S(r,f). \tag{7}$$

On the other hand, from (5) we see that any zero of u of order $n > k$ is either a pole of $c/(v-4b)$ with order not less than n or a pole of $1/(v-4b)$ with order $n-k$. Thus

$$\begin{aligned} N\left(r, \frac{1}{u}\right) &\le k\overline{N}\left(r, \frac{1}{u}\right) + N\left(r, \frac{c}{v-4b}\right) + N\left(r, \frac{1}{v-4b}\right) \\ &\le k\overline{N}(r, 1/\psi) + S(r,f) \end{aligned} \tag{8}$$

Combining (7) and (8) we get

$$\begin{aligned} T(r,f) &= T(r,u) + S(r,f) = T(r,1/u) + S(r,f) \\ &\le k\overline{N}(r, 1/\psi) + S(r,f). \end{aligned}$$

Thus we have

$$T(r,f) \le (2k+1)\,\overline{N}\left(r, 1\Big/\left(f^2 + af^{(k)} - b\right)\right) + S(r,f) \le S(r,f),$$

we arrive at a contradiction since f is trascendental. Theorem 1 is proved.

Remark. It is not difficult to prove that Theorem 1 still holds if we substituting $a(z)$ for a, where $a(z)(\neq 0)$ is a small function of f.

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