Lacunary Almost Summability in Certain Linear Topological Spaces

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Abstract. In this paper, the concept of lacunary almost summability of sequences in locally convex spaces has been defined and investigated. It is also proved Kojima-Schur and Silvarman-Toeplitz type theorems for lacunary almost conservatively and lacunary almost regularity of the which transform sequences in a Frechet space into a sequence in an other Frechet space. We also stated that the same results hold in locally bounded linear topological spaces.

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1. Introduction and background

Lacunary summability or lacunary almost summability of sequences of real (or complex) number by infinite matrices of real (or complex) numbers were given in [4].

In this paper, our purpose is to extend the concept of lacunary almost summability to two type of linear topological spaces Frechet spaces and locally baunded spaces, each of them includes Banach spaces as a special cases.

By a lacunary sequences we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ [3].

We consider only Hausdorff spaces. Throught this paper we use (X, p_i) and (Y, q_j) to denote Frechet spaces, i.e., locally convex spaces which are metrizable and complete, whose topologies are generated respectively by the countable clases $\{p_i\}$ and $\{q_i\}$ of seminorms.

If X and Y are linear topological spaces and Y is locally convex (with the topology generated by the semi-norms q_i), than the topology of bounded convergence on the spaces $L(X, Y)$ of continuous linear operators on X into Y is a locally convex topology generated by the seminorms $\{q_j^M\}$ where for each j and each bounded set M in X,

$$q_j^M(u) = \sup_{x \in M} q_j(u(x)), \quad u \in L(X, Y).$$

If (x_n) is a sequence in a Frechet spaces (X, p_i), the convergence of (x_n) in X will be in the topology considered, i.e, (x_n) is convergent to $x \in X$ if and only if $p_i(x_n - x) \to 0$ $(n \to \infty)$ for each i.

$\ell_\infty(X)$ and $c(X)$ will denote linear space of bounded and convergent sequences in X respectively.

2. Definitions

Definition 1. *Let (X, p_i) be a Frechet space (or locally convex space). A sequence $x = (x_n) \in \ell_\infty(X)$ is said to be lacunary almost convergent to l if and only if*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} x_{n+k} = l \quad uniformly \ in \ n.$$

Let the matrix $A = (A_{nk})$, $n, k = 0, 1, 2, \cdots$ consist of entries A_{nk} each of them is a continuous linear operator from a linear topological space X into another linear topological space Y.

We formally define the sequence (y_n),

$$y_n = \sum_{k=0}^{\infty} A_{nk} x_k \quad , \quad n = 0, 1, \cdots$$

as the A-transform of a given sequence (x_n) in X and write $y = Ax$. The matrix A is said to be conservative if for each convergent sequence (x_n) in X, its A-transform (y_n) is defined and is convergent in Y. The matrix A is said to be L-regular if for each $x_n \to x$ in X implies, $y_n \to L(x)$, where L is a prescribed continuous linear operator from X into Y. In particular, if we consider transformations of sequences in (X, p_i) into sequences in the same space then we may define regularity with respect to the identity operator I. In this case A is called to be regular matrix.

Definition 2. *Let X and Y be locally convex spaces. A sequence (x_n) in X is said to be lacunary almost A-summable if the A-transfrom of (x_n) is lacunary almost convergent in Y. The sequence (x_k) is said to be lacunary almost A-summable to $y \in Y$ if the A-transfrom of (x_k) is lacunary almost convergent to y in Y.*

Definition 3. *The matrix A is said to be lacunary almost conservative if the A-transform of (x_k) is lacunary almost convergent when $(x_k) \in c(X)$.*

Definition 4. *The matrix A is called to be lacunary almost L-regular if for each $x_k \to x$ in X, its A-transforms the sequences in X into the sequences in X then the identity operator I can be taken instead of L and A is said to be lacunary almost regular matrix.*

The following Lemmas will be used frequently in the proof of the theorems.

Lemma 1. [1, p. 22] *If X and Y are locally convex spaces and X is quasi-complete i.e. its closed and bounded sets of X are complete, then any collection of continuous linear operators from X into Y which is simply bounded is bounded for the topology of uniform convergence on bounded sets.*

Lemma 2. [2, p. 55] *Let (T_n) be a sequence of continuous linear operators on X into Y where X and Y are Frechet spaces. If $\lim_n T_n(x)$ exists for each x in a fundamental set in X and if for each $x \in X$, $(T_n x)$ is bounded in Y then $Tx = \lim_n T_n x$ exists for each $x \in X$ and T is a continuous linear operator on X into Y.*

3. Lacunary almost summability theorems for Frechet spaces

Theorem 1. *Let (X, p_i) and (Y, q_j) be Frechet spaces. The matrix $A = (A_{nk})$ defining sequence to sequence transformation from X into Y is lacunary almost conservative if and only if*

(i)   *for each bounded set M_α in X and for each fixed j,*

$$q_j \left(\sum_{k=0}^{m} \left(\frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \right) x_k \right) \leq K_{\alpha,j}$$

for $m, n, r = 0, 1, \cdots$ and $x_k \in M_\alpha$, $k = 0, 1, \cdots$,

(ii)   *for each $x \in X$, for each fixed r and for each n,*

$$\sum_{k=0}^{\infty} \left(\frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \right) x_k \quad exists$$

and

$$\lim_r \frac{1}{h_r} \sum_{k=0}^{\infty} \left(\frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \right) x_k \quad \textit{exists, uniformly in n;}$$

(iii)   *for each $x \in X$ and for each fixed $k = 0, 1, \cdots$*

$$\lim_r \frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \, x_k \quad \textit{exists, uniformly in n.}$$

Proof. First, suppose that A is lacunary almost conservative. Then for each $x = (x_k) \in c(X)$, the A-transform of (x_k) is defined and lacunary almost convergent in Y. To show that (i) is necessary, let us consider the linear space $c(X)$ of convergent sequences (x_k) in X. It is easily seen that P_i is a semi-norm on $c(X)$ for each i, where

$$P_i(x) = \sup_k p_i(x_k), \quad x = (x_k) \in c(X),$$

and also locally convex space $(c(X), P_i)$ is complete and so quasi complete, then for each fixed n, the collection $\{T_{nm} : n, \, m = 0, 1, 2, \cdots\}$ of linear operators defined by

$$T_{nm}x = \sum_{k=0}^{m} A_{nk} \, x_k, \quad x = (x_k) \in c(X),$$

are continuous on $c(X)$ into Y. By the same reasons, for each fixed n and r, the collection $\{U_{n,r,m} : m = 0, 1, 2, \cdots\}$ of linear operators defined by

$$U_{n,r,m} \, x = \frac{1}{h_r} \sum_{v \in I_r} T_{v+n,m} \, x = \frac{1}{h_r} \sum_{v \in I_r} \sum_{k=0}^{m} A_{v+n,m} \, x_k$$

$$= \sum_{k=0}^{m} \left(\frac{1}{h_r} \sum_{k \in I_r} A_{v+n,k} \right) x_k, \quad x = (x_k) \in c(X)$$

are continuous on $c(X)$ into Y. By the assumption of the existence of the A-transform of every $x = (x_k) \in c(X)$,

$$\lim_m U_{n,r,m} \, x = U_{n,r} \, x = y_{n,r}$$

exists for every fixed r and n. Thus, by Lemma 2, for $n, r = 0, 1, \cdots$, each $U_{n,r}$ is continuous and linear on $c(X)$ into Y. Since A is also lacunary almost conservative matrix and $x = (x_k) \in c(X)$, it follows that $(y_{n,r})_{r \geq 0}$ is convergent in (Y, q_j) uniformly in n and consequently for $n = 0, 1, 2, \cdots$, $(y_{n,r})_{r \geq 0} = (U_{n,r}x)_{r \geq 0}$ is bounded in (Y, q_j). Thus the collection $\{U_{n,r} : r = 0, 1, 2, \cdots\}$ of continuous linear operators is pointwise (and therefore simply) bounded for $n = 1, \cdots$. Therefore by Lemma 1, they are bounded convergence on $L(c(X), Y)$ for $n = 0, 1, \cdots$.

From the description of the topology of the bounded convergence we obtain

$$\sup_{x \in M} q_j(U_{n,r}\, x) < K_{M,j} \quad n, r = 0, 1, \cdots,$$

for each fixed j and each bounded set M in $c(X)$.

Now consider a bounded set M_α in X. This set consists of points x such that $p_i(x) < \alpha_i$. Consider the sequence of the from $(x_0, x_1, \cdots, x_n, 0, 0, \cdots)$ where x_k's are in M_α. All such sequences are above in the same bounded set M_α in $c(X)$. Hence, from the result we obtained we get

$$q_j\left(\sum_{k=0}^{m} \left[\frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \right] x_k \right) \leq K_{M_\alpha, j} = K_{\alpha, j}, \quad m, n, r = 0, 1, \cdots$$

for $x_k \in M_\alpha$ for each k. The proof of the necessity of condition (I) is complete.

The necessity of (ii) follows by considering the sequence $(x, x, x, \cdots)$ while that of (iii) follows by considering the sequence (x_n), $x_n = 0$ if $n \neq k$ and $x_k = x$.

For the sufficiency, let us consider the sequences of the form $(x, x, x, \cdots)$ and $(0, 0, 0, \cdots, 0, x_k, 0, 0, \cdots, 0, \cdots)$. It is easily seen that the set of the sequences of this from is a fundamental set in $c(X)$. Further, for each $x = (x_k) \in c(X)$ and for each $n, m = 0, 1, \cdots$, the operators T_{nm} are linear and continuous on $c(X)$ into Y. Hence the operators $U_{n,r,m}$ are also linear and continuous on $c(X)$ into Y. Then by the condition (i), for every fixed $x = (x_k) \in c(X)$ and fixed n and r, the sequence $(U_{n,r,m}\, x)_{m \geq 0}$ is bounded in Y. And also, by the condition (ii), for every fixed $x = (x_k)$ belonging to the fundamental set.

$$\lim_m U_{n,r,m}\, x, \quad n = 0, 1, \cdots$$

exists for every fixed r. Hence, for every fixed r and $n = 0, 1, \cdots$, the operators $U_{n,r}$ defined by

$$U_{n,r}\, x = \lim_m U_{n,r,m}\, x,$$

are linear and continuous an $c(X)$ into Y by Lemma 2. Further, for every $n = 0, 1, \cdots$ and for fixed $x = (x_k) \in c(X)$, the sequence $(U_{n,r}x)_{r \geq 0}$ is bounded and by (ii) and (iii) this sequence is convergent in Y for every x belonging to the fundamental set. Thus again by Lemma 2 for every $x = (x_k) \in c(X)$ and for $n = 0, 1, \cdots$ $\lim_r U_{n,r}x = U_n x$ exists and U_n ($n = 0, 1, \cdots$) is linear and continuous on $c(X)$ into Y.

Now it may be verified that under the conditions of the theorem, $U_n x$ can be written in the form of

$$U_r x = \lim_r \frac{1}{h_r} \sum_{V \in I_r} \sum_{k=0}^{\infty} A_{v+n,k}\left(\lim_\mu x_\mu \right) + \sum_{k=0}^{\infty} \lim_r \frac{1}{h_r} \sum_{V \in I_r} A_{v+n,k}\left(x_k - \lim_\mu x_\mu \right) \quad (*)$$

for each $x = (x_k) \in c(X)$ and $n = 0, 1, \cdots$.

It is also observed that the right hand side of (*) is independent on n by the conditions (ii) and (iiii), so that the limit $U_n x = \lim_r U_{n,r}x$ is uniform in n. This completes the proof of the theorem.

The following theorem for lacunary almost regularity is easily proved in a similar manner to that of the Theorem 1.

Theorem 2. *The matrix $A = (A_{nk})$ defining sequence to sequence transformations from the Frechet space (X, p_i) into the Frechet space (Y, q_j) is lacunary almost regular if and only if*

(a) of Theorem 1 hold and

(b) for each $x \in X$ for each fixed r and for each n

$$\sum_{k=0}^{\infty} \left(\frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k} \right) x_k \quad exists$$

and

$$\lim_r \frac{1}{h_r} \sum_{v \in I_r} \sum_{k=o} A_{v+n,k}\, x = Lx, \quad uniformly\ in\ n.$$

(c) for every $x \in X$ and for every fixed k,

$$\lim_r \frac{1}{h_r} \sum_{v \in I_r} A_{v+n,k}\, x = 0, \quad uniformly\ in\ n.$$

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