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Strong Convergence of Weighted Averaged Approximants of Asymptotically Nonexpansive Mappings in Banach Spaces without Uniform Convexity

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Abstract. Let C be a nonempty closed convex subset of a reflexive Banach space whose norm is uniformly Gâteaux differentiable, $T : C \to C$ an asymptotically nonexpansive mapping and P the sunny nonexpansive retraction from C onto $F(T)$. In the paper, we introduce property (S) for mapping T as minimal condition for strong convergence to Px of the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x + (1 - \alpha_n) \sum_{i=1}^{n} \alpha_{ni} T^i x_n, \quad n \geq N_0,$$

where $\{\alpha_n\}$ and $\{\alpha_{ni}\}$ $(i = 1, 2, \cdots)$ are real sequences satisfying appropriate conditions and N_0 is sufficiently large natural number.

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1. Introduction

Let C be a nonempty subset of a real Banach space E and let $T : C \to C$ be a nonlinear mapping. The mapping T is said to be *asymptotically nonexpansive* if for each $n \geq 1$, there exists a positive constant k_n with $\lim_{n\to\infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|$$

for all $x, y \in C$. T is nonexpansive if $k_n = 1$ for $n = 1, 2, \cdots$.

Let C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping such that the set $F(T)$ of fixed points of T is nonempty and x be an element of C. Browder [2] proved that $\{x_t\}$ defined by

$$x_t = tx + (1-t)\, Tx_t, \quad 0 < t < 1$$

converges strongly to an element of $F(T)$ which is nearest to x in $F(T)$ as $t \to 0$ in case when E is a Hilbert space. Reich [7] extended Browder's result in the framework of a uniformly smooth Banach space. Lim and Xu [6] partially extended celebrated convergence theorem of Reich [7] for asymptotically nonexpansive mappings in same framework.

Recently, using an idea of Browder [2], Shimizu and Takahashi [8] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = a_n x + (1-a_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n, \quad n = 1, 2, \cdots,$$

where $\{a_n\}$ is a real sequence satisfying $0 < a_n < 1$ and $a_n \to 0$. Shioji and Takahashi [9] and Jung et al. [5] extended their results in uniformly convex Banach spaces.

In this paper, we study existence and strong convergence of the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x + (1-\alpha_n)\sum_{i=1}^{n} a_{ni} T^i x_n, \quad n = 1, 2, \cdots, \tag{1}$$

where T is an asymptotically nonexpansive mapping on a nonempty closed convex subset of a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Our results improve previous known results in [5, 6, 7, 8, 9].

2. Preliminaries and Lemmas

Let E be a Banach space and E^* be the dual space of E. The value of $y \in E^*$ at $x \in E$ will be denoted by $\langle x, y \rangle$. We also denote by J, the duality mapping from E into E^*, that is,

$$J(x) = \left\{ y \in E^* : \langle x, y \rangle = \|x\|^2,\ \|x\| = \|y\| \right\} \quad \text{for each } x \in E.$$

Recall that a Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists for each x and y in $S = \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly *Gâteaux differentiable* if for each $y \in S$, this limit is attained uniformly for $x \in S$. The norm is said to be Fréchet differentiable if for each $x \in S$, this limit is attained uniformly for $y \in S$. Finally, the norm is said to be *uniformly Fréchet differentiable* if the limit is attained uniformly for $(x, y) \in S \times S$. In this case E is said to be *uniformly smooth*. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm.

Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ be an asymptotically nonexpansive mapping. Without loss of generality, we may assume that $k_n \geq 1$ for all $n = 1, 2, \cdots$. Set

$$A_n x := \sum_{i=1}^{n} a_{ni} T^i x, \qquad x \in C, \ n = 1, 2, \cdots,$$

where $\{a_{ni}\}$ $(i = 1, 2, \cdots)$ are sequences of real numbers such that

(i) $a_{ni} \geq 0$ for all $n = 1, 2, \cdots$,

(ii) $\sum_{i=1}^{n} a_{ni} = 1$,

(iii) $\sum_{i=1}^{n} a_{ni} k_i = \beta_n \geq 1$ for all $n = 1, 2, \cdots$ and $\lim_{n \to \infty} \beta_n = 1$.

Now, for an $x \in C$ and a positive integer n, consider a mapping T_n on C defined by

$$T_n u = \alpha_n x + (1 - \alpha_n) A_n u, \quad u \in C, \tag{2}$$

where $\{\alpha_n\}$ is a sequence of real numbers such that

$$0 < \alpha_n \leq 1, \ \lim_{n \to \infty} \alpha_n = 0, \text{ and } \limsup_{n \to \infty} \frac{\beta_n - 1}{\alpha_n} < 1.$$

Lemma 1. *Let* C *be a nonempty closed convex subset of a Banach space E and* $T : C \to C$ *be an asymptotically nonexpansive mapping. Let* T_n *be a mapping defined by* (2). *Then* T_n *has exactly one fixed point* x_n *in C such that*

$$x_n = \alpha_n x + \left(1 - \alpha_n\right) A_n x_n \quad \text{for all} \quad n \geq N_0, \tag{3}$$

where N_0 *is a sufficiently large natural number.*

Proof. Since $\limsup_{n\to\infty} \frac{\beta_n - 1}{\alpha_n} < 1$, there exists a natural number N_0 such that $(1 - \alpha_n)\beta_n < 1$ for all $n \geq N_0$. So, for each $n \geq N_0$, there exists the unique point x_n in *C* satisfying $x_n = \alpha_n x + (1 - \alpha_n) A_n x_n$, since the mapping T_n defined by (2) satisfies $\| T_n u - T_n v \| \leq (1 - \alpha_n)\beta_n \| u - v \|$ for all $u, v \in C$.

Lemma 2. *Let C, T and X be as in Lemma* 1. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *is bounded.*

Proof. Since $(1 - \alpha_n)\beta_n < 1$ for all $n \geq N_0$, for each $\varepsilon > 0$, there exists a natural number n_0 such that $\frac{\alpha_n}{1-(1-\alpha_n)\beta_n} < \varepsilon$ for all $n \geq n_0$. Then for $v \in F(T)$, we have

$$\begin{aligned} \| x_n - v \| &= \| \alpha_n x + (1 - \alpha_n) A_n x_n - v \| \\ &\leq \alpha_n \| x - v \| + \left(1 - \alpha_n\right) \sum_{i=1}^{n} a_{ni} \| T^i x_n - T^i v \| \end{aligned}$$

implies

$$\| x_n - v \| \leq \frac{\alpha_n}{1 - (1 - \alpha_n)\beta_n} \| x - v \| \leq \varepsilon \| x - v \|$$

for all $n \geq n_0$.

Let μ be a continuous linear functional on ℓ^∞ and let $(a_0, a_1, a_2, \cdots) \in \ell^\infty$. We write $\mu_n(a_n)$ instead of $\mu(a_0, a_1, a_2, \cdots)$. We call μ a Banach limit [1] when μ satisfies:

$$\| \mu \| = \mu_n(1) = 1 \text{ and } \mu_n(a_{n+1}) = \mu_n(a_n)$$

for all $(a_0, a_1, a_2, \cdots) \in \ell^\infty$. For a Banach limit μ, we know that

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for all $(a_0, a_1, a_2, \cdots) \in \ell^\infty$.

Let $\{x_n\}$ be a bounded sequence in E. Then we can define a real-valued continuous convex function on E by

$$f(z) = \mu_n \| x_n - z \|^2$$

for all $z \in E$. The following lemma is given in [3, 4, 10].

Lemma 3 [3, 4, 10]. *Let C be a nonempty closed convex subset of a Banach space whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded sequence in C, $u \in C$ and μ be a Banach limit. Then*

$$f(u) = \min_{z \in C} f(z)$$

if and only if

$$\mu_n \langle z - u, J(x_n - u) \rangle \leq 0$$

for all $z \in C$.

Let C be a convex subset of E, D a nonempty subset of C and P a retraction from C onto D, that is, $Px = x$ for all $x \in D$. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in C$ and $t \geq 0$. D is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. For more details, see [3].

The following lemma is well known (cf. [3]).

Lemma 4 [3]. *Let C be a convex subset of a smooth Banach space, D be a nonempty subset of C and P be a retraction form C onto D. Then P is sunny and nonexpansive if*

$$\langle x - Px, J(z - Px) \rangle \leq 0 \quad \textit{for all} \quad x \in C, \; z \in D.$$

Lemma 5. *Let C, T and X be as in Lemma* 1. *Then* $\mu_n \langle x_n - x, J(x_n - z) \rangle \leq \alpha \mu_n \| x_n - z \|^2$ *for all* $z \in F(T)$, *where* $\alpha = \limsup_{n \to \infty} \dfrac{\beta_n - 1}{\alpha_n} < 1.$

Proof. Since from (3)

$$\alpha_n (x_n - x) = (1 - \alpha_n) \; (A_n x_n - x_n),$$

we get for $z \in F(T)$ and for each $n \geq N_0$,

$$\langle x_n - x, J(x_n - z)\rangle = \frac{1-\alpha_n}{\alpha_n}\langle A_n x_n - x_n, J(x_n - z)\rangle$$
$$= \frac{1-\alpha_n}{\alpha_n}\langle A_n x_n - A_n z + z - x_n, J(x_n - z)\rangle$$
$$\le \frac{1-\alpha_n}{\alpha_n}(\beta_n - 1)\| x_n - z\|^2$$
$$\le \frac{\beta_n - 1}{\alpha_n}\| x_n - z\|^2$$

which gives

$$\mu_n\langle x_n - x, J(x_n - z)\rangle \le \alpha\mu_n\| x_n - z\|^2.$$

Finally, we introduce the following minimal property for convergence of sequence $\{x_n\}$ defined by (3):

Definition 1. *Let C be a nonempty closed convex subset of a Banach space E and* $T : C \to C$ *be a mapping. Then T is said to satisfy the property* (S) *if the following holds: for each bounded sequence* $\{x_n\}$ *in C,*

$$\lim_{n\to\infty}\| x_n - Tx_n\| = 0 \ \textit{ implies } M \cap F(T) \ne \phi. \tag{S}$$

where $M = \{u \in C : f(u) = \inf_{z\in C} f(z)\}$.

The examples of such mappings that satisfy the property (S) are nonexpansive mappings in uniformly smooth Banach space (see [7]).

3. Main results

Theorem 1. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable,* $T : C \to C$ *be an asymptotically nonexpansive mapping with Lipschitz constant* k_n *which satisfies the property* (S), *and P be the sunny nonexpansive retraction form C onto* $F(T)$. *Let* $\{a_{ni}\}$ $(i = 1, 2, \cdots\}$ *be real sequences satisfying:*

$$a_{ni} \ge 0, \ \sum_{i=1}^{n} a_{ni} = 1 \ \textit{ and } \ \sum_{i=1}^{n} a_{ni}k_i = \beta_n \ge 1, \quad n = 1, 2, \cdots.$$

Let $\{\alpha_n\}$ *be a real sequence such that*

$$0 \le \alpha_n \le 1, \quad \lim_{n\to\infty} \alpha_n = 0 \quad and \quad \lim_{n\to\infty} \sup \frac{\beta_n - 1}{\alpha_n} < 1.$$

Then

(a) *for any* $x \in C$, *there is exactly one* $x_n \in C$ *such that*

$$x_n = \alpha_n x + (1-\alpha_n)\sum_{i=1}^{n} a_{ni} T^i x_n \quad for\ all\ n \ge N_0,$$

where N_0 *is a sufficiently large natural number.*

(b) *If* $\{x_n\}$ *is an approximate fixed point sequence for* T, *i.e.,* $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$, *it follows that* $\{x_n\}$ *converges strongly to Px.*

Proof.

(a) The result follows from Lemma 1.

(b) From Lemma 2, it follows that $\{x_n\}$ is bounded. Define a real-valued function on E by

$$f(z) = \mu_n \| x_n - z \|^2$$

for all $z \in E$. Then, since f is continuous and convex, $f(z) \to \infty$ as $\| z \| \to \infty$ and E is reflexive, f attains its infimum over C. Let $u \in C$ such that $f(u) = \inf_{z\in C} f(z)$. Then $M = \{u \in C : f(u) = \inf_{z\in C} f(z)\}$ is nonempty because $u \in M$. Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| Tx_n - x_n \| = 0$, by property (S), T has a fixed point in M. Denote such a point y. It follows from Lemma 3 that

$$\mu_n \langle x - y,\ J(x_n - y) \rangle \le 0.$$

This inequality and Lemma 5 yields

$$\mu_n \| x_n - y \|^2 \le \alpha\mu_n \| x_n - y \|^2,$$

that is,

$$\mu_n \| x_n - y \|^2 \le 0.$$

Therefore, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to $y \in F(T)$. Then, by Lemma 5, we have

$$\langle y - x, J(y - Px) \rangle \leq \alpha \| y - Px \|^2 .$$

This inequality and Lemma 4 yield

$$\| y - Px \|^2 \leq \alpha \| y - Px \|^2.$$

From $\alpha < 1$, we have $y = Px$. To complete the proof, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ which converges strongly to z. We shall show that $y = z$. Since $y = Px$, it follows from Lemma 4 and 5 that

$$\langle x - Px, J(z - Px) \rangle \leq 0 \quad \text{and} \quad \langle z - x, J(z - Px) \rangle \leq \alpha \| z - Px \|^2 ,$$

and hence we have

$$\begin{aligned} \| y - z \|^2 &= \langle z - y, J(z - Px) \rangle \\ &= \langle z - x, J(z - Px) \rangle + \langle x - Px, J(z - Px) \rangle \\ &\leq \alpha \| y - z \|^2. \end{aligned}$$

Thus, $\{x_n\}$ converges strongly to Px.

Recall that a nonempty subset *D* of *C* is said to satisfy the Property (P) (cf. [6]) if the following holds:

$$x \in D \text{ implies } \omega_\omega(x) \subset D, \tag{P}$$

where $\omega_\omega(x)$ is the weak ω-limit set of *T*, that is, the set

$$\{ y \in C : y = weak - \lim_i T^{n_i} x \text{ for some } n_i \to \infty \} .$$

The following lemma is crucial to prove our next main result.

Lemma 6. *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C be a nonempty closed convex subset of E and* $T : C \to C$ *be an asymptotically nonexpansive mapping with Lipschitz constant* k_n. *Let* $\{x_n\}$ *be a bounded sequence with* $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. *Assume that every nonempty weakly* *compact convex subset of C satisfying the property* (P) *has a fixed point for T. Then T satisfies property* (S).

Proof. Note that $M = \{u \in C : f(u) = \inf_{z \in C} f(z)\}$ is nonempty closed convex bounded subset of *C*. Although *M* is not necessarily invariant under *T*, it does have the property (P). In fact, if $u \in M$ and $y = weak - \lim_{j \to \infty} T^{n_j} u$ belongs to the weak ω-limit set $\omega_\omega(u)$ of *T* at *u*, then from weak lower semicontinuity of *f* and $\lim_{n \to \infty} \| x_n - Tx_n \| = 0$, we have

$$\begin{aligned} f(y) &\le \liminf_{j \to \infty} f(T^{n_j} u) \le \limsup_{n \to \infty} f(T^n u) \\ &= \limsup_{n \to \infty} \left(\mu_m \| x_m - T^n u \|^2\right) = \limsup_{n \to \infty} \left(\mu_m \| T^n x_m - T^n u \|^2\right) \\ &\le \limsup_{n \to \infty} k_n^2 \mu_m \| x_m - u \|^2 = \mu_m \| x_m - u \|^2 \\ &= \inf_{z \in C} f(z). \end{aligned}$$

Thus, $y \in M$ and hence M satisfies the property (P). It follows from assumption that *T* has a fixed point in *M*. Therefore, *T* has satisfies property (S).

Theorem 2. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable, $T : C \to C$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n and P be the sunny nonexpansive retraction from C onto* F(T). *Let $\{a_{ni}\}$ $(i = 1, 2, \cdots)$ be real sequences satisfying:*

$$a_{ni} \ge 0, \sum_{i=1}^{n} a_{ni} = 1 \text{ and } \sum_{i=1}^{n} a_{ni} k_i = \beta_n \ge 1, \quad n = 1, 2, \cdots.$$

Let $\{\alpha_n\}$ be a real sequence such that

$$0 \le \alpha_n \le 1, \quad \alpha_n \to 0 \text{ and } \limsup_{n \to \infty} \frac{\beta_n - 1}{\alpha_n} < 1,$$

and let $\{x_n\}$ be a sequence in C defined by (3) *such that* $\lim_{n \to \infty} \| x_n - Tx_n \| = 0$. *Assume that every nonempty weakly compact convex subset of C satisfying the property* (P) *has a fixed point for T. Then $\{x_n\}$ converges strongly to Px.*

Proof. The result follows from Lemma 6 and Theorem 1.

In the case when $a_{ni} = \frac{1}{n}$ for $i = 1, 2, \cdots n$, in Theorem 1 and 2, we have the following corollaries.

Corollary 1. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable, $T : C \to C$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n which satisfies the property* (S), *and P be the sunny nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence such that*

$$0 \le \alpha_n \le 1, \lim_{n\to\infty} \alpha_n = 0 \text{ and } \limsup_{n\to\infty} \frac{\beta_n - 1}{\alpha_n} < 1, \text{ where } \beta_n = \tfrac{1}{n}\sum_{i=1}^{n} k_i .$$

Then

(a) For any $x \in C$, there is exactly one $x_n \in C$ such that

$$x_n = \alpha_n x + (1 - \alpha_n)\frac{1}{n}\sum_{i=1}^{n} T^i x_n \quad \text{for all } n \ge N_0, \tag{4}$$

where N_0 is a sufficiently large natural number.

(b) If $x_n - Tx_n \to 0$ as $n \to \infty$, it follows that $\{x_n\}$ converges strongly to Px.

Corollary 2. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable, $T : C \to C$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n and P be the sunny nonexpansive retraction from C onto $F(T)$. Let $\{\alpha_n\}$ be a real sequence such that*

$$0 \le \alpha_n \le 1, \alpha_n \to 0 \text{ and } \limsup_{n\to\infty} \frac{\beta_n - 1}{\alpha_n} < 1, \text{ where } \beta_n = \frac{1}{n}\sum_{i=1}^{n} k_i .$$

and let $\{x_n\}$ be a sequence in C defined by (4) *such that $\lim_{n\to\infty}\| x_n - Tx_n \| = 0$. Assume that every nonempty weakly compact convex subset of C satisfying the property* (P) *has a fixed point for T. Then $\{x_n\}$ converges strongly to Px.*

In the case when $a_{nn} = 1$ for all n and $a_{ni} = 0$ otherwise, we have the following:

Corollary 3. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable, $T : C \to C$ be an asymptotically nonexpansive mapping with Lipschitz constant k_n which satisfies the property* (S), *and*

P be the sunny nonexpansive retraction from C onto $F(T)$. *Let* $\{\alpha_n\}$ *be a real sequence in* $(0, 1)$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{k_n - 1}{\alpha_n} < 1$$

and let $\{x_n\}$ *be a sequence in C defined by*

$$x_n = \alpha_n x + (1-\alpha_n) T^n x_n, \qquad n \geq N_0.$$

Suppose in addition that $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. *Then* $\{x_n\}$ *converges strongly to* Px.

Remark. (1) Corollary 1 and 2 extend Theorem 1 of [5] and Theorem 2 of [9] from uniformly convex Banach space to reflexive Banach space, respectively.
(2) Corollary 3 improves Theorem 2 of Lim and Xu [6], where the space is assumed to be uniformly smooth.

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