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Coincidence Theorems for Contractive Type Multivalued Mappings

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Abstract. In this paper we prove some coincidence theorems for contractive multivalued mappings on a compact metric space. Our results extend properly the corresponding results of Hu and Rosen [1] and Rao [2].

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1. Introduction

Let (X, d) be a metric space, N be the set of all positive integers. We denote by $CL(X), CB(X)$ and $C(X)$ the families of all nonempty closed, nonempty closed bounded, nonempty compact subsets of X, respectively, and by H the Hausdorff metric on $CB(X)$ induced by the metric d on X. That is,

$$H(A, B) = \max\left\{\sup_{a\in A} D(a, B),\ \sup_{b\in B} D(b, A)\right\} \quad \text{for} \quad A, B \in CB(X),$$

where $D(a, B) = \inf_{b\in B} d(a, b)$. It is obvious that $CL(X) = CB(X) = C(X)$ if (X, d) is a compact metric space. For $A, B \in CB(X)$, let

$$\delta(A, B) = \sup_{a\in A, b\in B} d(a, b) \quad \text{and} \quad \delta(A) = \delta(A, A).$$

Let $f : X \to X$ be a single valued mapping, T and $G : X \to CL(X)$ be multivalued mappings. f and G are said to be commutative or strongly commutative if $fGx \subseteq Gfx$ or $Gfx \subseteq fGx$ for all $x \in X$. The composition of G and T is defined by

$$TGx = T(Gx) = \bigcup_{y \in Gx} Ty \quad \text{for} \quad x \in X.$$

A point z in X is said to be a coincidence point of f and G if $fz \in Gz$ and a fixed point of G if $z \in Gz$.

Hu and Rosen [1] established a fixed point theorem for multivalued mappings G satisfying

$$H(Gx, Gy) < d(x, y) \tag{1.1}$$

for all $x, y \in X$ with $x \neq y$.

Rao [2] obtained coincidence theorems for multivalued mappings G and single valued mappings f, which satisfy the following condition

$$H(Gx, Gy) < \max\left\{d(fx, fy), D(fx, Gx), D(fy, Gy), \frac{1}{2}\left[D(fx, Gy) + D(fy, Gx)\right]\right\} \tag{1.2}$$

for all $x, y \in X$ with $fx \neq fy, Gx \neq Gy, fx \notin Gx, fy \notin Gy$.

The main purpose of this paper is to investigate the existence of coincidence point for multivalued mappings G and single valued mappings f which satisfy the following condition

$$H(Gx, Gy) < \max\left\{d(fx, fy), D(fx, Gx), D(fy, Gy), \frac{1}{2}\left[D(fx, Gy) + D(fy, Gx)\right], \frac{D(fx, Gx)D(fy, Gy)}{d(fx, fy)}, \frac{D(fx, Gy)D(fy, Gx)}{d(fx, fy)}\right\} \tag{1.3}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \notin Gx$, $fy \notin Gy$. Our results extend properly the corresponding results of Hu and Rosen [1] and Rao [2].

2. Coincidence theorems

In this section we assume that (X, d) is a compact metric space and $G : X \to CL(X)$ is a multivalued mapping and f is a single valued mapping of X. We need the following.

Lemma 2.1. [1]. *Let all powers of* G *map* X *into* $CL(X)$ *and* G^m *be continuous for some* m *in* N. *Let* $A = \bigcap_{n \in N} G^n X$. *Then* A *is a nonempty compact subset of* X *and* $GA = A$.

Theorem 2.2. *Let all powers of* fG *map* X *into* $CL(X)$ *and* f, G, *and* $(fG)^m$ *be continuous, where* m *is some element in* N. *Suppose that*

$$f \text{ and } G \text{ are commutative,} \tag{2.1}$$

$$G(fG)x \subseteq (fG)Gx \text{ for all } x \text{ in } X, \tag{2.2}$$

and (1.3) *holds. Then* f *and* G *have a coincidence point in* X.

Proof. Let $A = \bigcap_{n \in N} (fG)^n X$. By Lemma 2.1, we obtain that A is a nonempty compact subset of X and $fGA = A$. Now we claim that for all $x \in X$,

$$f(fG)^n x \subseteq (fG)^n fx, \quad n \in N; \tag{2.3}$$

$$G(fG)^n x \subseteq (fG)^n Gx, \quad n \in N. \tag{2.4}$$

It follows from (2.1) that (2.3) holds for $n = 1$. Suppose that (2.3) holds for some $n \in N$. Then

$$\begin{aligned} f(fG)^{n+1} x &= \left[f(fG)^n \right] fGx \subseteq \left[(fG)^n f \right] fGx \\ &= (fG)^n f(fGx) \subseteq (fG)^n fGfx = (fG)^{n+1} fx. \end{aligned}$$

That is, (2.3) holds for $n + 1$. By induction, we infer that (2.3) holds. Similarly, we can prove that (2.4) holds. In view of (2.3) and (2.4), we conclude that

$$fA = f \bigcap_{n \in N} (fG)^n X \subseteq \bigcap_{n \in N} f(fG)^n X \subseteq \bigcap_{n \in N} (fG)^n fX \subseteq A,$$

and

$$GA = G \bigcap_{n \in N} (fG)^n X \subseteq \bigcap_{n \in N} G(fG)^n X \subseteq \bigcap_{n \in N} (fG)^n GX \subseteq A.$$

Consequently $A = fGA \subseteq fA$ and $A = fGA \subseteq GfA \subseteq GA$. Hence $A = fA = GA$. By the continuity of f and G, we know that $D(fz, Gz) = \inf\{D(fx, Gx) \mid x \in A\}$ for some z in A. Since $Gz \in CL(X)$, there exists $y \in Gz$ with $D(fz, Gz) = d(fz, y)$. From $fA = GA = A$, we can find $w \in A$ with $fw = y$. Suppose that $fz \neq fw$, $Gz \neq Gw$, $fz \notin Gz$, $fw \notin Gw$. Note that

$$0 < d(fz, fw) = D(fz, Gz) \leq D(fw, Gw) \leq H(Gz, Gw). \tag{2.5}$$

Using (1.3) and (2.5), we get that

$$\begin{aligned} H(Gz, Gw) &< \max\Big\{d(fz, fw), D(fz, Gz), D(fw, Gw), \frac{1}{2}\left[D(fz, Gw) + D(fw, Gz)\right], \\ &\qquad \frac{D(fz, Gz)\, D(fw, Gw)}{d(fz, fw)}, \frac{D(fz, Gw)\, D(fw, Gz)}{d(fz, fw)}\Big\} \\ &= \max\Big\{D(fz, Gz), D(fz, Gz), D(fw, Gw), \frac{1}{2} D(fz, Gw), D(fw, Gw), 0\Big\} \\ &= \max\Big\{D(fw, Gw), \frac{1}{2}\left[H(Gz, Gw) + D(fz, Gz)\right]\Big\} \\ &\leq H(Gz, Gw), \end{aligned}$$

which is a contradiction and hence either $fz = fw \in Gz$ or $fw \in Gw = Gz$ or $fz \in Gz$ or $fw \in Gw$. This means that $fz \in Gz$ by (2.5). This completes the proof.

Similarly we have

Theorem 2.3. *Let all powers of fG map X into $CL(X)$ and $(fG)^m$ be continuous for some $m \in N$. Let f and G satisfy (2.1) and (2.2) and*

$$\delta(Gx, Gy) < \delta\left(\bigcup_{n,k=0}^{\infty}\left[f^n G^k\{x, y\} \cup G^k f^n\{x, y\}\right]\right) \tag{2.6}$$

for all x, y in X with $Gx \neq Gy$. Then f and G have a unique common fixed point z in X. Further $Gz = \{z\}$ and $z = fz$.

Proof. Let $A = \bigcap_{n \in N} (fG)^n X$. As in the proof of Theorem 2.2, we deduce that $A = fA = GA$. By Lemma 2.1, we see that A is a nonempty compact subset. Suppose that $\delta(A) > 0$. Then there exist a, b in A with $\delta(A) = d(a, b)$. Note that $A = GA$. There exist $x, y \in A$ with $a \in Gx, b \in Gy$. Clearly $\delta(A) = \delta(Gx, Gy)$ and $fx, fy \in A$. By virtue of (2.6), we get that

$$\delta(A) = \delta\,(Gx, Gy) < \delta\left(\bigcup_{n,k=0}^{\infty}\left[\, f^n G^k\{x, y\} \cup G^k f^n\{x, y\}\right]\right) \leq \delta\,(A),$$

which is impossible and hence $\delta(A) = 0$. That is, A is a singleton set, say, $A = \{z\}$ for some z in X. It is obvious that $fz = z$ and $Gz = \{z\}$.

Suppose that f and G have a second common fixed point w. Then $w \in (fG)^n X$ for all $n \in N$ and hence $w \in A = \{z\}$. That is, f and G have a unique common fixed point z in X. This completes the proof.

Theorem 2.4. *Let* f *and* G *be continuous, strongly commutative and satisfy* (1.3). *Then* f *and* G *have a coincidence point in* X.

Proof. Let $A = \bigcap_{n \in N} f^n X$. It follows from Lemma 2.1 that A is a nonempty compact subset of X and $fA = A$. Since f and G is strongly commutative, we infer that

$$Gf^n x = Gf\,(f^{n-1}x) \subseteq fGf^{n-1}x \subseteq \cdots \subseteq f^n Gx, \tag{2.7}$$

for all $x \in X$ and $n \in N$. In view of (2.7), we conclude that

$$GA = G \bigcap_{n \in N} f^n X \subseteq \bigcap_{n \in N} Gf^n X \subseteq \bigcap_{n \in N} f^n GX \subseteq A.$$

The rest of the result follows as in Theorem 2.2. This completes the proof.

Remark 2.1 The following example verifies that Theorem 2.4 does indeed generalize Theorem 3.2 of Hu and Rosen [1] and Theorem 3 and Theorem 4 of Rao [2].

Example 2.1. Let $X = \{3, 4, 5, 7\}$ with the usual metric. Take $f = i$ — the identity mapping. Define a mapping $G : X \to CL(X)$ by $G3 = \{3, 5\}, G4 = \{7\}, G5 = \{3\}$, and $G7 = \{4, 5, 7\}$. Suppose that x, y are in X with $x \neq y, Gx \neq Gy$, $x \notin Gx, y \notin Gy$. Then (x, y) is in $\{(4, 5), (5, 4)\}$. It is easy to see that

$$H(Gx, Gy) = 4 < 6 = \max\left\{d(fx, fy), D(fx, Gx), D(fy, Gy), \frac{1}{2}[D(fx, Gy) + D(fy, Gx)],\right.$$
$$\left.\frac{D(fx, Gx)\,D(fy, Gy)}{d(fx, fy)}, \frac{D(fx, Gy)\,D(fy, Gx)}{d(fx, fy)}\right\}$$

for $(x, y) \in \{(4, 5), (5, 4)\}$. It is easy to check that all conditions of Theorem 2.4 are satisfied. But Theorem 3.2 of Hu and Rosen [1] and Theorem 3 and Theorem 4 of Rao [2] are not applicable since

$$H(Gx, Gy) = 4 > 3 = \max\left\{d(fx, fy), D(fx, Gx), D(fy, Gy),\right.$$
$$\left.\frac{1}{2}[D(fx, Gy) + D(fy, Gx)]\right\}$$

for $x = 4$ and $y = 5$.

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