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Some Results on Anti-Invariant Submanifolds of a Trans-Sasakian Manifold

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Abstract. In [7] Oubina introduced a new class of almost contact metric structure known as trans-Sasakian structure which is a generalization of both α-Sasakian and β-Kenmotsu structures [6]. The geometry of anti-invariant submanifolds of Sasakian manifolds have been investigated by Yano and Kon and many others [9,10] etc.

On the other hand, anti-invariant submanifolds of Kenmotsu manifold have been studied by the present author [5]. The purpose of this paper is to study anti-invariant submanifolds of trans-Sasakian manifold generalizing some results on the above mentioned topics.

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1. Introduction

Let $\overline{M}^{2n+1}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g) where ϕ is a tensor field of typo (1.1), ξ a vector field, η a 1-form and g is the Riemannian metric on $\overline{M}$. Then these tensors satisfy [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi) \tag{1.2}$$

for any vector fields X, Y tangent to $\overline{M}$.

An almost contact structure (ϕ, ξ, η) is called normal if the almost complex structure J on $\overline{M} \times R$ given by

$$J\left[X, f\frac{d}{dt}\right] = \left[\phi X - f\xi, \qquad \eta(X)\frac{d}{dt}\right]$$

f being a $\subset^{\infty}$ function on $\overline{M} \times R$, is integrable, or equivalently

$[\phi, \phi] + 2d\eta \otimes \xi = 0$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

According to Gray and Hervella [4], in the classification of almost Hermitian manifolds, there appears, a class of Hermitian manifolds namely ω_4 which contains locally conformal Kahler manifold. An almost contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-Sasakian if $(\overline{M} \times R, J, g)$ belongs to the class ω_4 where g is a Riemannian metric on $\overline{M} \times R$. This may be expressed by the condition [2]

$$\left(\overline{\nabla}_X \phi\right)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi(X)\} \tag{1.3}$$

for functions α and β on $\overline{M}$ and the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$, and in this case we say that the trans-Sasakian structure is of type (α, β).

If $\alpha = 0$ then $\overline{M}$ is a β-Kenmotsu manifold and if $\beta = 0$ then $\overline{M}$ is α-Sasakian manifold [6]. Moreover if $\alpha = 1$ and $\beta = 0$ then $\overline{M}$ is a Sasakian manifold and if $\alpha = 0$ and $\beta = 1$ then $\overline{M}$ is a Kenmotsu manifold. From (1.3), it follows that

$$\overline{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{1.4}$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in a trans-Sasakian manifold $\overline{M}$. We denote by g the metric tensor on $\overline{M}$ as well as that induced on M. Let T_xM and $T_x^{\perp}M$ denote the tangent and normal bundles of M at $x \in M$. Let ∇ and $\overline{\nabla}$ denote the covariant differentiation with respect to the metrics on M and $\overline{M}$, respectively. The Gauss and Weingarten formulae for M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{1.5}$$

respectively, where h is the second fundamental form of M in $\overline{M}$, and $\nabla^{\perp}$ is the operator of covenant differentiation with respect to the linear connection induced in the normal bundle $T_X^{\perp}M$.

Moreover,

$$g\left(h(X,Y), N\right) = g(A_N X, Y).$$

A submanifold M of a trans-Sasakian manifold $\overline{M}$ is called invariant if $\phi\, T_X M \subset T_X M$. On the other hand, if $\phi\, T_X M \subset T_X^{\perp} M$ for all $x \in M$, then M is said to be anti-invariant in $\overline{M}$.

Now suppose M^m is an m-dimensional anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}^{2n+1}$. Then for every vector $\overline{Z}$ of $\overline{M}^{2n+1}$ at a point of M^m, we put

$$\overline{Z} = \overline{Z}_t + \overline{Z}_n \tag{1.6}$$

where $\overline{Z}_t$ and $\overline{Z}_n$ are tangential and normal vectors to M^m, respectively. Define homomorphisms P and Q of the normal bundle into the tangent and normal bundles of M^m respectively, by

$$PN = (\phi N)_t \quad \text{and} \quad QN = (\phi N)_n \tag{1.7}$$

for every normal vector field N of M^m.

If X is a vector field on an anti-invariant submanifold M^m of a trans-Sasakian manifold $\overline{M}^{2n+1}$, then ϕX is a vector field in the normal bundle of M^m, where $m > 1$, as any 1-dimensional submanifold is anti-invariant.

Now operating ϕ on ϕX, ϕN and ξ and comparing tangential and normal components, we get the following :

$$\begin{aligned} &-X + \eta(X)\xi_t = P\phi X, \qquad \eta(X)\xi_n = Q\phi X, \\ &\eta(N)\xi_t = PQN, \qquad -N + \eta(N)\xi_n = \phi PN + Q^2 N, \\ &\phi\xi_t + P\xi_n = 0, \qquad Q\xi_n = 0 \end{aligned} \tag{1.8}$$

for any $X \in TM$ and $N \in T^{\perp}M$.

2. Anti-invariant submainfold of trans-Sasakian manifold when ξ is tangent to M

In what follows we assume that ξ is tangent to M^m. Then $\xi_n = 0$ and (1.8) becomes

$$\begin{aligned} &-X + \eta(X)\xi = P\phi X, \qquad Q\phi X = 0, \\ &PQN = 0, \qquad -N = \phi PN + Q^2 N. \end{aligned} \tag{2.1}$$

From (2.1), we find that $Q^3 + Q = 0$, and hence, Q defines an f-structure in the normal bundle [8].

We now assume that M^m is an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}^{2n+1}$. Then differentiating ϕX, ϕN, and ξ in the direction of a tangent vector field on M^m and using (1.3), (1.6) and Gauss and Weingarten formulae, we have the following lemmas.

Lemma 2.1. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is tangent to M. Then*

$$A_{QX}Y + Ph(X,Y) = \alpha\left[\eta(X)Y - g(X,Y)\xi\right], \tag{2.2}$$

$$\nabla_Y^{\perp}\phi X - Qh(X,Y) - \phi\overline{\nabla}_Y X = -\beta\,\eta(X)\,\phi Y \tag{2.3}$$

for any $X, Y \in T$.

Lemma 2.2. *Let M be an anti-invariant submanifold of trans-Sasakian manifold $\overline{M}$ such that ξ is tangent to M. Then*

$$P\nabla_X^{\perp}N = \overline{\nabla}_X PN + A_{QN}X, \tag{2.4}$$

$$-QA_N X + Q\nabla_X^{\perp}N = h(X,PN) + \nabla_X^{\perp} QN \tag{2.5}$$

for any $X \in TM$ and $N \in T^{\perp}M$.

Lemma 2.3. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}^{2n+1}$ such that ξ is tangent to M. Then*

$$\nabla_X\xi = \beta(X - \eta(X)\xi), \tag{2.6}$$

$$h(X,\xi) = \alpha\phi X \tag{2.7}$$

for all $X \in TM$.

From the above lemmas, as particular cases, we have

Lemma 2.4. *Let M be an anti-invariant submanifold of a α-Sasakian manifold $\overline{M}$ such that ξ is tangent to M. Then*

$$A_{QX}Y + Ph(X,Y) = \alpha\left[\eta(X)Y - g(X,Y)\xi\right],$$

$$\nabla_Y^{\perp}\phi X - Qh(X,Y) = \phi\overline{\nabla}_Y X$$

for any $X, Y \in TM$.

Lemma 2.5. *Let M be an anti-invariant submanifold of a β-Kenmotsu manifold $\overline{M}$ such that ξ is tangent to M. Then*

$$A_{QX}Y + Ph(X,Y) = 0,$$

$$\nabla^{\perp}_{Y}\phi X - Qh(X,Y) = \phi \overline{\nabla}_{Y} X - \beta\eta(X)\phi Y$$

for any $X, Y \in TM$.

Lemma 2.6. *Let M be an anti-invariant submanifold of a α-Sasakian $\overline{M}$ such that ξ is tangent to M. Then*

$$\nabla_{X}\xi = 0, \qquad h(X,\xi) = \alpha\phi X$$

for all $X \in TM$.

Lemma 2.7. *Let M be an anti-invariant submanifold of a β-Kenmotsu manifold $\overline{M}$ such that ξ is tangent to M. Then*

$$\nabla_{X}\xi = \beta(X - \eta(X)\xi),$$

$$h(X,\xi) = 0$$

for all $X \in TM$.

Now from Lemma 2.2. we have

$$\overline{\nabla}_{\xi}PN - P\nabla^{\perp}_{\xi}N = A_{QN}\xi.$$

Also

$$g(A_{QN}\xi, X) = g(h(X,\xi), QN) = \alpha g(\phi X, QN) = 0.$$

Thus

$$(\overline{\nabla}_{\xi}P)(N) = \overline{\nabla}_{\xi}PN - P\nabla^{\perp}_{\xi}N = 0,$$

and similarly

$$(\overline{\nabla}_{\xi}Q)(N) = \nabla^{\perp}_{\xi}QN - Q\nabla^{\perp}_{\xi}N = 0.$$

Hence, we have

Proposition 2.9. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is tangent to M. Then*

(a) ξ is parallel vector field along M and $h(\xi,\xi)$ vanishes in the direction of ξ.

(b) P and Q are parallel along ξ.

Now, suppose that $\dim M = m = n+1$. Then $Q = 0$ and from Lemma 2.2, we have

$$\overline{R}(X,Y)PN = PR^{\perp}(X,Y)N,$$

for vector field X, Y tangent to M, where $R^{\perp}$ is the curvature tensor on the normal bundle. Thus $\overline{R} = 0$ implies that $R^{\perp} = 0$. Conversely, if $R^{\perp} = 0$ then $\overline{R}(X,Y)PN = 0$ and also $\overline{R}(X,Y)\xi = 0$. Hence $\overline{R}(X,Y) = 0$.

Thus we have

Proposition 2.9. *Let $\overline{M}$ be a $(2n+1)$-dimensional trans-Sasakian manifold and M^{n+1} be an anti-invariant submanifold $\overline{M}^{2n+1}$ with ξ tangent to M^{n+1}. Then $\overline{R} = 0$ if and only if $R^{\perp} = 0$.*

Next, we prove

Proposition 2.10. *Let M^{n+1} be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}^{2n+1}$ such that ξ is tangent to M^{n+1}. Then M cannot be totally umbilical when $n \geq 1$.*

Proof. Suppose M is totally umbilical. Then $h(X,Y) = g(X,Y)H$, where H is the mean curvature vector. From (2.7), we have $h(\xi,\xi) = 0$ which implies that $g(\xi,\xi)H = 0$ and therefore M is minimal and hence totally geodesic. Thus, we have $h(X,\xi) = 0$ and consequently $\alpha\phi X = 0$, which is a contradiction as $n > 1$. Hence M is not totally umbilical, whereby proving the result.

Proposition 2.11. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ with ξ tangent to M^m. Then we have*

$$(\overline{\nabla}_X F)(X,\xi) = -\alpha, \tag{2.8}$$

$$(\overline{\nabla}_X \eta)(X) = \beta \tag{2.9}$$

where F is the fundamental 2-form given by

$$F(X,Y) = g(X,\phi Y).$$

Proof. From (1.3) and (1.4), we have

$$(\overline{\nabla}_X F)(Y,Z) = -\alpha\{g(X,Z)\eta(Y) - g(X,Y)\eta(Z)\} - \beta\{g(X,\phi Z)\eta(Y) - g(X,\phi Y)\eta(Z)\}$$

and

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X,Y) - \beta\{g(X,Y) - \eta(X)\eta(Y)\}$$

so that our assertion follows from the above equations.

Proposition 2.12. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ with ξ tangent to M. If $A_N X = 0$ for any $N \in T_X^{\perp}M$ then $\phi(T_X M)$ is parallel with respect to the normal connection.*

Proof. Using Gauss and Weingarten formulae and Equation (1.3), we have

$$\begin{aligned}\nabla_X^{\perp}\phi Y &= \overline{\nabla}_X \phi Y + A_{\phi Y}X = (\overline{\nabla}_X \phi)(Y) + \phi\overline{\nabla}_X Y + A_{\phi Y}X \\ &= \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\} + \phi\overline{\nabla}_X Y + A_{\phi Y}X.\end{aligned}$$

Since $A_N = 0$ for any $N \in T^{\perp}M$, we have

$$\begin{aligned}g(\nabla_X^{\perp}\phi Y, N) &= -\beta\eta(Y)g(\phi X, N) - g(\overline{\nabla}_X Y, \phi N) \\ &= -\beta\eta(Y)g(\phi X, N) + g(\phi\nabla_X Y, N) + g(\phi h(X,Y), N) \\ &= \beta\eta(Y)g(X,\phi N) - g(\nabla_X Y, \phi N) - g(h(X,Y),\phi N) \\ &= -g(A_{\phi N}X, Y) = 0\end{aligned}$$

as $\phi N \in T^{\perp}M$ for any $N \in T^{\perp}M$, which proves the result.

3. Anti-invariant submanifold of trans-Sasakian manifold when ξ is normal to M

In this section, we assume that ξ is normal to M^m. Then $\xi_t = 0$ and from (1.8), we get

$$-X = P\phi X, \quad Q\phi X = 0, \quad PQN = 0,$$
$$-N + \eta(N)\xi = \phi PN + Q^2 N$$

for any $X \in TM$, $N \in T^{\perp}M$.

Now suppose that $\overline{M}^{2n+1}$ is a trans-Sasakian manifold. Then differentiating ϕX, ϕN and ξ covariantly and using (1.3), (1.7) and Gauss and Weingarten formulae, we have the following:

Lemma 3.1. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M. Then*

$$A_{\phi Y}X = Ph(X,Y), \tag{3.1}$$

$$\nabla^{\perp}_X \phi Y = -\alpha g(X,Y)\xi + \phi\overline{\nabla}_X Y + Qh(X,Y) \tag{3.2}$$

for any $X, Y \in TM$.

Lemma 3.2. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M. Then*

$$\nabla_X PN - A_{\phi N}X - P\nabla^{\perp}_X N + \alpha\eta(N)X = 0, \tag{3.3}$$

$$h(X, PN) + \nabla^{\perp}_X QN - Q\nabla^{\perp}_X N + \phi A_N X = \beta g(\phi X, N)\xi - \eta(N)\phi X \tag{3.4}$$

for any $X \in TM$, $N \in T^{\perp}M$.

Lemma 3.3. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M. Then*

$$-A_{\xi}X = \beta X, \tag{3.5}$$

$$\nabla^{\perp}_X \xi = -\alpha\phi X \tag{3.6}$$

for $X \in TM$.

We now prove the following.

Proposition 3.4. *If M is an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M, then the curvature tensor of the normal bundle annihilates ξ.*

Proof. From (3.2) and (3.6), we get

$$\begin{aligned}\nabla_Y^{\perp}(\nabla_X^{\perp}\xi) &= \nabla_Y^{\perp}(-\alpha\phi X) = -\alpha(\nabla_Y^{\perp}\phi\ X)\\ &= -\alpha\left\{-\alpha g(X,Y)\xi + \alpha\overline{\nabla}_Y X + Qh(X,Y)\right\}\\ &= \alpha^2 g(X,Y)\xi - \alpha\phi\overline{\nabla}_Y X - \alpha Qh(X,Y)\end{aligned}$$

for $X, Y \in TM$.

Now,

$$\begin{aligned}R^{\perp}(X,Y)\xi &= \nabla_X^{\perp}\nabla_Y^{\perp}\xi - \nabla_Y^{\perp}\nabla_X^{\perp}\xi - \nabla_{[X,y]}^{\perp}\xi\\ &= \nabla_X^{\perp}(-\alpha\phi Y) - \nabla_Y^{\perp}(-\alpha\phi X) - \alpha\phi[X,Y]\\ &= -\alpha\left\{\nabla_X^{\perp}(\phi Y) - \nabla_Y^{\perp}(\phi X) - \phi[X,Y]\right\}\end{aligned}$$

which, in view of (3.2), gives that

$$R^{\perp}(X,Y)\xi = 0$$

whereby proving the result.

Suppose that $m = n$ and hence $Q = 0$. Then from (3.1)–(3.6), we have

Lemma 3.5. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M. Then*

$$A_{\phi Y}X = \phi h(X,Y), \quad \nabla_X^{\perp}(\phi Y) = -\alpha g(X,Y) + \phi\overline{\nabla}_X Y, \tag{3.7}$$

$$\alpha\eta(N)X - \phi\nabla_X^{\perp}N = \nabla_X\phi N, \tag{3.8}$$

$$A_{\xi}X = -\beta X, \qquad \nabla_X^{\perp}\xi = -\alpha\phi X \tag{3.9}$$

for any $X, Y \in TM$.

Proposition 3.6. *Let M be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that ξ is normal to M. Then the connection in the normal bundle is trivial if and only if M is of constant curvature $-\alpha^2$.*

Proof. Using $\nabla_X^{\perp}(\phi Y) = -\alpha g(X,Y) + \phi\overline{\nabla}_X Y$ and $\nabla_X^{\perp}\xi = -\alpha\phi X$

we have

$$R^{\perp}(X,Y)\phi Z = \nabla_X^{\perp}\left(\nabla_Y^{\perp}\phi Z\right) - \nabla_Y^{\perp}\left(\nabla_X^{\perp}\phi Z\right) - \nabla_{[X,Y]}^{\perp}\phi Z$$

$$= -\alpha g(Y,Z)\nabla_X^{\perp}\xi + \overline{\nabla}_X^{\perp}(\phi\overline{\nabla}_Y Z) + \alpha g(X,Z)\nabla_Y^{\perp}\xi$$

$$- \nabla_Y^{\perp}(\phi\overline{\nabla}_X Z) + \alpha g([X,Y],Z)\xi - \phi\overline{\nabla}_{[X,Y]}Z$$

or,

$$R^{\perp}(X,Y)\phi Z = \alpha^2 g(Y,Z)\phi X + \nabla_X^{\perp}\left(\phi\nabla_Y^{\perp}Z\right) - \alpha^2 g(X,Z)\phi Y$$

$$- \nabla_Y^{\perp}\left(\phi\overline{\nabla}_X Z\right) + \alpha g([X,Y],Z)\xi - \phi\overline{\nabla}_{[X,Y]}Z. \tag{3.10}$$

Also,

$$\phi\overline{R}(X,Y)Z + \alpha^2\{g(Y,Z)\phi X - g(X,Z)\phi Y\}$$

$$= \nabla_X^{\perp}(\phi\overline{\nabla}_Y Z) + \alpha g(X,\overline{\nabla}_Y Z)\xi - \nabla_Y^{\perp}(\phi\overline{\nabla}_X Z) - \alpha g(Y,\overline{\nabla}_X Z)\xi$$

$$- \phi\overline{\nabla}_{[X,Y]}Z + \alpha^2\{g(Y,Z)\phi X - g(X,Z)\phi Y\}$$

which on further simplification, gives

$$\phi\overline{R}(X,Y)Z + \alpha^2\{g(Y,Z)\phi X - g(X,Z)\phi Y\} = R^{\perp}(X,Y)\phi Z \tag{3.11}$$

for any $X, Y, Z \in TM$.

From (3.11) we find that if the connection of the normal bundle is trivial, that is, $R^{\perp} \equiv 0$. Then M^n is of constant curvature $-\alpha^2$.

Conversely, if M^n is of constant curvature $-\alpha^2$, then from (3.11) we have $R^{\perp}(X,Y)\phi Z = 0$. Moreover, from Proposition 3.4, we have $R^{\perp}(X,Y)\xi = 0$, which completes the proof.

From the above result, we have

Corollary 3.7. *Let M be an anti-invariant submanifold of a β-kenmotsu manifold $\overline{M}$ such that ξ is normal to M. Then the connection in the normal bundle is trivial if and only if M is of zero constant curvature.*

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