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Connectedness in Monotone Spaces

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Abstract. In this paper connectedness is studied in weaker (monotone, quasi-directed) spaces and we examine sets having the ending property (e.p.).

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1. Introduction

In this paper we attempt to extend the notion of connectedness in a monotone (neighbourhood) space [3,7] which is weaker than closure (directed neighbourhood) space [3,4]. Due to the weaker axiom (the monotone property) in a monotone space with comparison to the stronger axiom (the closure preserving property) in a closure space, we observe that the making of comparable extensions face serious problems in obtaining corresponding basic results. To overcome these difficulties and for existing need for further extension, we introduce the notion of a *q*-directed space. Finally, to complete our study we further introduce and study the ending property of a set, quasi-nodal set, separating point, end point, ending and external boundary point in monotone spaces, weaker than the space considered in [1].

2. Known definitions and theorems

Definition 2.1. [7]. *A function u from the power set $P(X)$ of a set X to $P(X)$ is called a monotone operator on X provided the following conditions are satisfied:*

(i) $u\phi = \phi$

(ii) $A \subset uA$ *for every* $A \in P(X)$ *and*

(iii) $A \subset B \Rightarrow uA \subset uB$ *for every* $A, \ B \in P(X)$.

A structure (X, u) where X is a set and u is a monotone operator on X is called a monotone space [7] (a neighbourhood space [3]) and for $A \subset X$, uA is called the monotone closure or simply closure (when there is no possibility of confusion) of A.

Definition 2.2. [7]. *Let (X, u) be a monotone space and let $A \subset X$. Then A is called closed if $uA = A$, and A is called open if $X - A$ is closed.*

Definition 2.3. [5,7]. *Let (X, u) be a monotone space and $A \subset X$. Then the interior of A, denoted by $\text{Int}\, A$, is a set defined by $\text{Int}\, A = X - u(X - A)$.*

Definition 2.4. [3,7]. *In a monotone space (X, u) a neighbourhood of a subset A of X is a subset U of X such that $A \subset \text{Int}\, U$. A neighbourhood of a point $x \in X$ is a neighbourhood of the one-point set $\{x\}$. The neighbourhood system of a set $A \subset X$ (of a point $x \in X$) is the collection of all neighbourhoods of the set A (of the point x).*

Definition 2.5. [5]. *In a monotone space (X, u), a cluster point or an accumulation point of a set $A \subset X$ is a point x belonging to the closure of $A - \{x\}$. A cluster point or an accumulation point of the space (X, u) is defined to be a cluster point of the underlying set X.*

Definition 2.6. [7]. *A monotone space (X, u) is said to be T_1 if $x \neq y$ implies $x \notin u(\{y\})$ and $y \notin u(\{x\})$.*

Definition 2.7. [3]. *Let (X, u) and (Y, v) be two monotone spaces and f a function from X to Y. Then f is said to be continuous at the point x of X iff the inverse, under f, of every neighbourhood of $f(x)$ is a neighbourhood of x. f is said to be continuous iff f is continuous at each point of X.*

Theorem 2.1. [5]. *Arbitrary intersection (union) of closed (open) sets in a monotone space is closed (open).*

Remark 2.1. [5]. Union (Intersection) of two closed (open) sets in a monotone space need not be closed (open).

Theorem 2.2. [5]. *Let (X, u) be a monotone space. Then*

(i) $\text{Int}\, X = X$, $\text{Int}\, \phi = \phi$
(ii) $\text{Int}\, A \subset A$ *for* $A \subset X$,
(iii) $\text{Int}\,(A \cap B) \subset \text{Int}\, A \cap \text{Int}\, B$ *for* $A, B \subset X$,
(iv) $A \subset B \Rightarrow \text{Int}\, A \subset \text{Int}\, B$ *for* $A, B \subset X$.

Theorem 2.3. [5]. *In a monotone space* (X, u),

(i) $uA = X - \text{Int}(X - A)$ *for any* $A \subset X$,

(ii) *a set* $A \subset X$ *is open iff* $\text{Int}\, A = A$.

Theorem 2.4. [5]. *In a monotone space* (X, u), *a subset U of X is a neighbourhood of a subset A of X iff U is a neighbourhood of each point of A.*

Theorem 2.5. [5]. *Let* (X, u) *be a monotone space. A subset U of X is open iff it is a neighbourhood of all of its points, or equivalently, U is a neighbourhood of itself.*

Theorem 2.6. [5]. *In a monotone space* (X, u), *a point* $x \in X$ *belongs to the closure of a subset A of X iff each neighbourhood of x in* (X, u) *intersects A.*

Theorem 2.7. [5]. *Let* (X, u) *and* (Y, v) *be two monotone spaces and* f *a function from X to Y. Let* $x \in X$. *Then each of the following conditions is necessary and sufficient for* f *to be continuous at x:*

(a) *for each neighbourhood V of* $f(x)$ *there exists a neighbourhood U of x such that* $f(U) \subset V$.

(b) *for every* $A \subset X$, $x \in uA$ *implies* $f(x) \in vf(A)$.

Theorem 2.8. [5]. *Let* (X, u) *and* (Y, v) *be two monotone spaces and* f *a function from X to Y. Then each of the following conditions is necessary and sufficient for* f *to be continuous:*

(a) *for each x in X, the inverse image of every neighbourhood of* $f(x)$ *is a neighbourhood of x.*

(b) $f(uA) \subset vf(A)$ *for every* $A \subset X$.

(c) $uf^{-1}(B) \subset f^{-1}(vB)$ *for every* $B \subset Y$.

Remark 2.2. Monotone spaces are generalization of topological spaces, because if (X, τ) is a topological space then the closure operator u of this topological space enjoys all the properties of a monotone operator and hence the closure operator u of the topological space (X, τ) makes X into a monotone space, viz., (X, u).

But not all monotone spaces are topological spaces which we explain in the following example:

We consider the monotone space (X, u) where $X = \{a, b, c\}$ and u: $P(X) \to P(X)$ is defined by $u\phi = \phi, uX = X$, $u\{a\} = \{a\}$, $u\{b\} = \{b\}$, $u\{c\} = u\{a, b\} = u\{b, c\} = u\{c, a\} = X$. Clearly, {a} and {b} are both closed in

(X, u), but $\{a,b\}$ is not closed in (X, u) as $u\{a,b\} = X \neq \{a,b\}$. Hence u is not a Kuratowski closure operator of a topological space and so, u cannot induce X into a topological space.

3. Some remarks on continuity on monotone spaces

Remark 3.1. We cannot replace the term 'neighbourhood' by 'open set' in Theorem 2.7 (a) as shown in the following example.

Example 3.1. Let $X = \{a,b,c\}$, $Y = \{x,y,z\}$. Then $P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}\}$ and $P(Y) = \{\phi, Y, \{x\}, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{z,x\}\}$. We now define $u : P(X) \to P(X)$ and $v : P(Y) \to P(Y)$ respectively by $u\phi = \phi$, $uX = X$, $u\{a\} = \{a\}, u\{b\} = \{b\}$, $u\{c\} = \{b,c\}, u\{a,b\} = u\{b,c\} = u\{c,a\} = X$ and $v\phi = \phi$, $vY = Y$, $v\{y\} = \{y\}, v\{x\} = v\{z\} = v\{x,y\} = v\{y,z\} = v\{z,x\} = Y$. Clearly, (X, u) and (Y, v) are monotone spaces. Now, we consider the mapping $f : X \to Y$ as follows:

$$f(a) = x,\ f(b) = z,\ f(c) = y.$$

Clearly, the neighbourhoods of the point 'a' are the sets $\{a,c\}, \{a,b\}, X$; but the open sets containing 'a' are the sets $\{a,c\}$ and X. Again, clearly the neighbourhoods of the point 'x' are the sets $\{x,z\}$ and Y; but the open sets containing 'x' are the sets $\{x,z\}$ and Y. Now, for the neighbourhoods of $x = f(a)$ we have the neighbourhood $\{a,b\}$ of a such that $f\{a,b\} = \{x,z\}$ is contained in both of $\{x,z\}$ and Y. So, f is continuous at a. But $\{x,z\}$ is an open set containing x and since $f\{a,c\} = \{x,y\}$ and $f(X) = Y$, hence there exists no open set U containing a such that $f(U) \subset \{x,z\}$.

However we have the following theorem:

Theorem 3.1. *If a mapping $f : (X,u) \to (Y,v)$ is continuous, then the inverse image of each open (closed) subset of Y is an open (closed) subset of X.*

Remark 3.2. The converse of Theorem 3.1 is not true, in general, as shown in the following example.

Example 3.2 . We consider the monotone space (X,u) of Example 3.1 and the set $Y = \{x,y,z\}$. We define $w : P(Y) \to P(Y)$ by $w\phi = \phi$, $wY = Y$, $w\{z\} = \{y,z\}$, $w\{y\} = \{y\}$ and $w\{x\} = w\{x,y\} = w\{y,z\} = w\{z,x\} = Y$. Clearly, (Y,w) is a monotone space. Now, we consider the mapping $f : X \to Y$ as follows: $f(a) = x, f(b) = f(c) = z$. The open sets in (X,u) and (Y,w) respectively are

X, ϕ, $\{b, c\}, \{c, a\}$; and Y, ϕ, $\{x, z\}$. Clearly, the inverse image of every open set of Y is open in X.

Now the neighbourhoods containing a are $\{a, c\}$, $\{a, b\}$, X and $f(X) = f\{a, c\} = f\{a, b\} = \{x, z\}$. Clearly, $\{x, y\}$ is a neighbourhood of x for which there is no neighbourhood U of a such that $f(U) \subset \{x, y\}$ and so, f is not continuous at x. Hence f is not continuous.

The continuity of the composite mapping is studied in the following theorem:

Theorem 3.2. *Let (X, u), (Y, v), (Z, w) be monotone spaces. Let $f : X \to Y$, $g : Y \to Z$ be continuous mappings. Then the composition mapping $h : X \to Z$, defined by $h(x) = g(f(x))$ for each x of X is continuous.*

4. Subspace

Definition 4.1. [7]. *Let X be a set and (X, u), (X, v) be two monotone spaces. Then u is said to coarser than v, and v to be finer than u, if $vA \subset uA$ for each $A \in P(X)$.*

Definition 4.2. [7]. *Let (X, u_X) be a monotone space and let $Y \subset X$. The monotone operator u_Y on Y is defined as $u_Y(A) = Y \cap u_X(A)$ for every $A \subset Y$. Then u_Y is called the relativization of u_X to Y and the space (Y, u_Y) is called the subspace of (X, u_X).*

The relativization of a closure u_X for a set X to a subset Y of X is the coarsest closure v for Y such that the identity mapping J_Y of (Y, v) into (X, u_X) is continuous. More precisely:

Theorem 4.1. *Let (X, u_X) be a monotone space and let $Y \subset X$. A closure v for Y is the relativization of u_X to Y if and only if the following two conditions are fulfilled:*

- *(a) the mapping $J_Y : (Y, v) \to (X, u_X)$ is continuous ;*
- *(b) If w is a closure for Y such that $J_Y : (Y, w) \to (X, u_X)$ is continuous, then w is finer than v.*

Proof.
Necessity. First suppose that v is the relativization of u_X to Y, i.e., $v(A) = Y \cap u_X(A)$ for each $A \subset Y$. The condition (a) is fulfilled because $J_Y(v(A)) = v(A) = Y \cap u_X(A) \subset u_X(A) = u_X(J_Y(A))$ for each $A \subset Y$. Now, if

$J_Y : (Y, w) \to (X, u_X)$ is continuous, then by definition, for each $A \subset Y$, $J_Y(w(A)) \subset u_X(J_Y(A))$, i.e., $w(A) \subset u_X(A)$; but $w(A) \subset Y$ and hence $w(A) \subset u_X(A) \cap Y = v(A)$ which shows that w is finer than v.

Sufficiency. Let v be a closure for Y satisfying (a) and (b). The relativization u_Y of u_X to Y fulfills (a) and (b) by necessity. From (b) we obtain that v is finer than u_Y and u_Y is finer than v which implies $v = u_Y$.

From the definition of subspace the following is an immediate consequence.

Theorem 4.2. *Let (Y, u_Y) be a subspace of (X, u_X) and (Z, v) be a monotone space such that $Z \subset Y$. Then (Z, v) is a subspace of (Y, u_Y) if and only if it is a subspace of (X, u_X).*

Proof. The proof follows from the following equalities:

$$Z \cap u_Y = Z \cap (Y \cap u_X) = Z \cap u_X .$$

Theorem 4.3. *Let (Y, u_Y) be a subspace of a monotone space (X, u_X). Then*

(a) if A is closed in X, $A \cap Y$ is closed in Y
(b) if A is open in X, $A \cap Y$ is open in Y
(c) if Y is closed in X and A is closed in Y, A is closed in X.

Proof. The proof is obvious.

Remark 4.1. If Y is closed-open in X and A is open in Y then A may not be either closed or open in X. Also, if Y is closed-open in X and A is closed in Y then A may not be open in X. We show it in the following example.

Example 4.1. Let $X = \{a, b, c\}$. We define $u : P(X) \to P(X)$ by $u(\phi) = \phi$, $u(X) = X$, $u(\{a\}) = u(\{a, b\}) = \{a, b\}$, $u(\{c\}) = \{c\}$, $u(\{b\}) = \{b\}$ and $u(\{b, c\}) = u(\{c, a\}) = X$. Clearly, (X, u) is a monotone space. Now, let $Y = \{a, b\}$. Since {c} is closed in X, so Y is closed-open in X. Also, v, the relativization of u is: $v(\phi) = \phi$, $v(Y) = v(\{a\}) = Y$ and $v(\{b\}) = \{b\}$. Clearly, $A = \{a\}$ is open in Y, but A is neither open nor closed in X. Again, if we take $A = \{b\}$ then A is closed in Y; but, clearly, A is not open in X.

Remark 4.2. If Y is open in X and A is closed in Y then A may not be closed in X. We show it in the following example.

Example 4.2. Let $X = \{a, b, c\}$. We define $u : P(X) \to P(X)$ by $u(\phi) = \phi$, $u(X) = X$, $u(\{a\}) = \{a, b\}$, $u(\{b\}) = \{b, c\}$, $u(\{c\}) = \{c\}$ and $u(\{a, b\}) = u(b, c\})$ $= u(\{c, a\}) = X$. Clearly, (X, u) is a monotone space. Now, let $Y = \{a, b\}$. Clearly, Y is open in X. Also, v, the relativization of u is : $v(\phi) = \phi$, $v(Y) = v(\{a\}) = Y$ and $v(\{b\}) = \{b\}$. Clearly, $A = \{b\}$ is closed in Y but A is not closed in X.

However, we have the following result as a corollary of Theorem 4.3(c).

Corollary 4.1. *Let (Y, u_Y) be a subspace of a monotone space (X, u_X). If Y is closed in X and A is open in Y, then $(X - Y) \cup A$ is open in X.*

Remark 4.3. Let (Y, u_Y) be a subspace of (X, u_X). Then a relatively closed (open) set need not be the intersection of Y and a closed (open) set in X. We show it in the following example.

Example 4.3. Let $X = \{a, b, c\}$. We define $u : P(X) \to P(X)$ by $u(\phi) = \phi$, $u(X) = X$, $u(\{a\}) = \{a\}$, $u(\{b\}) = \{a, b\}$, $u(\{c\}) = \{b, c\}$, $u(\{a, b\}) = u(\{b, c\})$ $= u(\{c, a\}) = X$. Let $A = \{b\}$ and $Y = \{b, c\}$. Since $u_Y(A) = u(A) \cap Y = \{b\}$, A is closed in Y. Now, ϕ, X and $\{a\}$ are the closed sets in X; but none of their intersection with Y yields A.

In the following theorem, we characterize a subspace of a monotone space.

Theorem 4.4. *Let (Y, v) and (X, u) be monotone spaces such that $Y \subset X$. Each of the following conditions is necessary and sufficient for (Y, v) to be a subspace of (X, u) :*

(a) *for each* $A \subset Y$, $\mathrm{Int}_v(A) = Y \cap \mathrm{Int}_u(A \cup (X - Y))$;

(b) *if $x \in Y$, then a set $V \subset Y$ is a neighbourhood of x in (Y, v) if and only if there exists a neighbourhood U of x in (x, u) such that $U \cap Y = V$.*

Proof. The steps of the proof will be: (a) is necessary, (a) implies (b) and finally (b) is sufficient.

Suppose that (Y, v) is a subspace of (X, u) and $A \subset Y$. By definition of the interior operation we have $\mathrm{Int}_u(A \cup (X - Y)) = X - u\ (X - (A \cup (X - Y)))$ $= X - u(Y - A)$ and consequently, $Y \cap \mathrm{Int}_u(A \cup (X - Y)) = Y \cap (X - u(Y - A))$ $= Y \cap (X \cap (X - u(Y - A))) = Y \cap (X - u(Y - A)) = Y - u(Y - A) = Y - (Y \cap$ $u(Y - A)) = Y - v(Y - A) = \mathrm{Int}_v(A)$. So, (a) is necessary.

Now, suppose (a). If V is a neighbourhood of an $x \in Y$ in (Y, v), then by (a) the set $U = V \cup (X - Y)$ is a neighbourhood of x in (X, u). Also, $U \cap Y = V$. Conversely, if U is a neighbourhood of x in (X, u) then also, the set $U \cup (X - Y)$ is a neighbourhood of x in (X, u) and by (a) the set $(U \cup (X - Y)) \cap Y = U \cap Y$ is a neighbourhood of x in (Y, v). Hence (a) implies (b).

Finally, suppose (b). Let $x \in Y$ and $A \subset Y$. If $x \in v(A)$ then each neighbourhood of x in (Y, v) intersects A, by Theorem 2.6. Let U be a neighbourhood of x in (X, u). Then by (b), $U \cap Y = V$ is a neighbourhood of x in (Y, v). So, V intersects A and hence U intersects A. Consequently, by Theorem 2.6, $x \in u(A)$. Therefore, $v(A) \subset u(A) \cap Y$. Conversely, if $x \in u(A)$ then by Theorem 2.6, each neighbourhood of x in (X, u) intersects A. Let V be any neighbourhood of x in (Y, v). Then by (b), there exists a neighbourhood U of x in (X, u) such that $U \cap Y = V$. So, U intersects A and hence V intersects A as $A \subset Y$. Consequently, $x \in v(A)$. Therefore, $u(A) \cap Y \subset v(A)$. So, $v(A) = u(A) \cap Y$. Hence (b) is sufficient. This proves the theorem.

We conclude this section with the following theorem which speaks of the continuity on a subspace.

Theorem 4.5. *Let (Y, u_Y) be a subspace of a monotone space (X, u_X) and let (Z, v) be a monotone space. If $f : (X, u_X) \to (Z, v)$ is continuous then $f_Y : (Y, u_Y) \to (Z, v)$ is also continuous, where f_Y is the restriction of f to Y.*

5. Connectedness

Definition 5.1. [5]. *Let (X, u) be a monotone space. Two subsets A and B of X are said to be semi-separated if there exist neighbourhoods U of A and V of B such that $U \cap B = \phi = V \cap A$.*

Theorem 5.1. [5]. *Let (X, u) be a monotone space. Then A_1, A_2 are two semi-separated subsets of X if and only if $(A_1 \cap uA_2) \cup (A_2 \cap uA_1) = \phi$.*

Theorem 5.2. [5]. *Let (X, u) and (Y, v) be monotone spaces and let $f : X \to Y$ be a continuous mapping. If A and B are semi-separated in (Y, v), then $f^{-1}(A)$ and $f^{-1}(B)$ possess the corresponding property in (X, u).*

Definition 5.2. *In a monotone space (X, u), the boundary of a subset A of X, denoted by $b(A)$, is a set defined by $b(A) = u(A) - \text{Int}\, A$, or equivalently, $b(A) = u(A) \cap u(X - A)$.*

Definition 5.3. *A subset A of a monotone space (X, u) is said to be connected in X if A is not the union of two non-void semi-separated subsets of X, that is* $A = A_1 \cup A_2, (uA_1 \cap A_2) \cup (A_1 \cap uA_2) = \phi$ *implies that* $A_1 = \phi$ *or* $A_2 = \phi$.

If A is not connected in *X* we say *A* is disconnected in *X*. A space (X, u) is said to be connected if the underlying set *X* is connected in (X, u). Every accrete space is connected and a non-void discrete space is connected if and only if it's underlying set is a singleton. If *A* is the union of two non-void semi-separated subsets of *X* then we say that A has a semi-separation.

Theorem 5.3. *A monotone space (X, u) is connected if and only if X is not the union of two disjoint, non-void, open subsets, that is, X contains no proper non-void subsets simultaneously open and closed.*

Proof. The proof is easy and hence omitted.

Theorem 5.4. *A monotone space (X, u) is connected if and only if X is not the union of two disjoint, non-void, closed subsets of X.*

Proof. The proof is obvious.

Theorem 5.5. *Let (Y, u_Y) be a subspace of a monotone space (X, u) and let $A \subset Y \subset X$. Then A is connected in (X, u) if and only if A is connected in (Y, u_Y).*

Proof. Let A be disconnected in *Y*. Then $A = A_1 \cup A_2$, $A_1 \neq \phi$, $A_2 \neq \phi$, $u_Y(A_1) \cap A_2 = \phi = A_1 \cap u_Y(A_2)$. Now, $u_Y(A_1) \cap A_2 = \phi$ implies $u(A_1) \cap Y \cap A_2 = \phi$, i.e., $u(A_1) \cap A_2 = \phi$. Similarly, $u(A_2) \cap A_1 = \phi$. So, *A* has a semi-separation in *X* and hence, *A* is disconnected in *X*.

Conversely, let A be disconnected in *X*. Then $A = A_1 \cup A_2, A_1 \neq \phi, A_2 \neq \phi$, $u(A_1) \cap A_2 = \phi = A_1 \cap u(A_2)$. Now $u_Y(A_1) \cap A_2 = u(A_1) \cap Y \cap A_2 = u(A_1) \cap A_2 = \phi$. Similarly, $u_Y(A_2) \cap A_1 = \phi$. So, A is disconnected in *Y*. This completes the proof of the theorem.

Corollary 5.1. *A subset A of a monotone space (X, u) is connected if and only if the subspace (A, u_A) is connected.*

Theorem 5.6. *A monotone space (X, u) is connected if and only if it contains no set A such that $\phi \neq A \neq X$ and $u(A) \cap u(X - A) = \phi$, i.e., every subset A of X such that $\phi \neq A \neq X$ satisfies the condition $u(A) \cap u(X - A) \neq \phi$; in other words, if and only if no set A, which satisfies condition $\phi \neq A \neq X$ has an empty boundary.*

Proof. The proof is omitted.

Theorem 5.7. *A subset C of a monotone space* (X, u) *is connected if and only if every subset A of C such that* $\phi \neq A \neq C$ *satisfies the condition* $C \cap u(A) \cap u(C - A) \neq \phi$.

Proof. The proof is obvious.

Theorem 5.8. *If C is connected and* $C \cap A \neq \phi \neq C - A$ *then* $C \cap b(A) \neq \phi$.

Proof. The proof is easy and hence omitted.

Theorem 5.9. *Let* (X, u) *and* (Y, v) *be two monotone spaces and* $f : (X, u) \to (Y, v)$ *be continuous. If A is a connected subset of X then* $f(A)$ *is connected in Y.*

Proof. If $f(A)$ is not connected then $f(A) = Y_1 \cup Y_2$, where Y_1 and Y_2 are semi-separated in (Y, v) and $Y_1 \neq \phi$, $Y_2 \neq \phi$. Since f is continuous, $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ are semi-separated in (X, u) by Theorem 5.2. Clearly, $A = f^{-1}(Y_1) \cup f^{-1}(Y_2)$ and $f^{-1}(Y_1) \neq \phi$, $f^{-1}(Y_2) \neq \phi$. Thus $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ form a semi-separation of *A*. Hence *A* is not connected in (X, u), which is a contradiction. Hence $f(A)$ is connected in (Y, v). This proves the theorem.

Theorem 5.10. *A subset C of a monotone space* (X, u) *is connected if and only if the following condition is fulfilled:*

If *C* is contained in the union of two semi-separated sets *A* and *B*, then $C \subset A$ or $C \subset B$.

Proof. Before going to prove the theorem, we first state the following lemma without proof.

Lemma. *Let* $C \subset A \cup B$ *and A, B are semi-separated. Then the sets* $C \cap A$ *and* $C \cap B$ *are also semi-separated.*

Proof of the theorem. We suppose that $C \subset A$ or $C \subset B$. We prove that *C* is connected. If *C* is not connected, then there exist semi-separated sets *A* and *B* such that $C = A \cup B$ where $A \neq \phi$, $B \neq \phi$, $A \cap u(B) = \phi = u(A) \cap B$. Clearly, $C \subset A$ (or $C \subset B$) is false, because then $B = \phi$ (or $A = \phi$). This contradicts our hypothesis that $C \subset A$ or $C \subset B$. Thus *C* is connected.

Conversely, we suppose that C is connected. If $C \subset A \cup B$ and if the sets A and B are semi-separated then by the lemma, the set $C \cap A$ and $C \cap B$ are also semi-separated and consequently, $C \cap A = \phi$ or $C \cap B = \phi$. That is, either $C \subset B$ or $C \subset A$. This completes the proof of the theorem.

Theorem 5.11. *Let $\{C_t\}$ be a family of connected sets in (X, u). The union $\bigcup_t C_t$ is connected, provided that there exists such a set C_0 which is not semi-separated from any set C_t.*

Proof. Let $\bigcup_t C_t = M \cup N$ where M and N are semi-separated. We are going to show that either $M = \phi$ or $N = \phi$. According to Theorem 5.10, we assume $C_0 \subset N$. Now we assert that for any t, $C_t \subset N$, because if for any t, $C_t \subset M$ then $u(C_t) \cap C_0 \subset u(M) \cap N = \phi$ and $C_t \cap u(C_0) \subset M \cap u(N) = \phi$ since M and N are semi-separated; and so, C_t and C_0 are semi-separated, a contradiction. Hence $M = \phi$. This completes the proof of the theorem.

Theorem 5.12. *Let $\{C_t\}$ be a directed family of connected sets (this means that for each pair t_1, t_2 there is t_3 such that $C_{t_1} \subset C_{t_2}$ and $C_{t_2} \subset C_{t_3}$). Then the union $S = \bigcup_t C_t$ is connected.*

Proof. Suppose $S = M \cup N$ where M and N are semi-separated sets. By Theorem 5.10, we have for each t, either $C_t \subset M$ or $C_t \subset N$. Let $C_{t_0} \neq \phi$. Obviously, we may assume that $C_{t_0} \subset M$; hence $C_{t_0} \not\subset N$. We shall show that $S \subset M$, which will complete the proof.

Let t be an arbitrary index and t_1 be such that $C_{t_0} \subset C_{t_1}$ and $C_t \subset C_{t_1}$. The first inclusion yields $C_{t_1} \not\subset N$. Hence $C_{t_1} \subset M$ and therefore $C_t \subset M$. It follows that $S \subset M$. This completes the proof of the theorem.

Corollary 5.2. *The union of connected sets, which have a non-empty intersection, is a connected set.*

Corollary 5.3. *If C is connected and $C \subset E \subset u(C)$, E is connected. In particular, $u(C)$ is connected.*

Corollary 5.4. *If every two points of a set E in a monotone space (X, u) are contained in some connected subset of E, then E is a connected set.*

6. Component

Definition 6.1. *Let (X, u) be a monotone space and $Y \subset X$. Then a set $C \subset Y$ is said to be a component of Y if C is connected in X and if the inclusion $C \subset C_1 \subset Y$ implies $C = C_1$ for any connected set C_1 in X.*

Clearly, C is a component of a monotone space (X, u) if C is a component of the underlying set X. Thus components are maximal connected subsets.

Theorem 6.1. *Let (X, u) be a monotone space and $Y \subset X$. Then a set $C \subset Y$ is a component of Y if and only if C is a component of the subspace (Y, u_Y).*

Proof. The proof is obvious.

Definition 6.2. *The component of a point x in (X, u) is the component of X containing x.*

Theorem 6.2. *Let (X, u) be a monotone space. Then*

(i) *Every component of (X, u) is closed.*
(ii) *Each point in X is contained in exactly one component of X.*
(iii) *The components of X form a partition of X, i.e., any two components are either disjoint or identical and union of all the components is X.*
(iv) *Each connected subset of X is contained in exactly one component of X.*

Proof. The proof is easy and hence omitted.

Theorem 6.3. *Every closed-open set F is the union of a family of components of the space (X, u). In particular, if F is connected and non-empty, then it is a component.*

Proof. Let $F \neq \phi$. Let $x \in F$. Then $\{x\}$ is a connected set contained in F. So, by Theorem 6.2. (iv), there is one and only one component $C(X, x)$ containing x. Clearly, $C(X, x) \cap F \neq \phi$. We assert that $C(X, x) - F = \phi$, because $C(X, x) - F \neq \phi$ gives $C(X, x) = (C(X, x) \cap F) \cup (C(X, x) - F)$ where $C(X, x) \cap F$ and $C(X, x) - F$ are closed in X (for, $C(X, x)$ is closed by Theorem 6.2 (i) and F is closed-open), forms a semi-separation of $C(X, x)$ contradicting the fact that $C(X, x)$ is connected. Hence, $x \in C(X, x) = C(X, x) \cap F \subset F$ which implies

$$\bigcup_{x \in F} \{x\} \subset \bigcup_{x \in F} C(X, x) \subset F, \text{ i.e., } F = \bigcup_{x \in F} C(X, x).$$

In particular, if F is connected and non-empty then F is a component, because there is no connected set C of which F is a proper subset, otherwise, $C - F$ and F form a semi-separation of C. This completes the proof of the theorem.

Definition 6.3. *Let (X, u) and (Y, v) be monotone spaces. A continuous mapping $f : X \to Y$ is called monotone if the inverse image $f^{-1}(C)$ of each connected set $C \subset Y$ is connected.*

Theorem 6.4. *Let $f : X \to Y$ be a monotone onto mapping. Then the set C is a component of $D \subset Y$ if and only if $f^{-1}(C)$ is a component of $f^{-1}(D)$.*

Proof. Since f is onto, $f^{-1}(C) \subset E \subset f^{-1}(D)$ implies $C \subset f(E) \subset D$. Now, if C is supposed to be a component of D and E is supposed to be connected, it follows that $C = f(E)$, hence $f^{-1}(C) = f^{-1}f(E) \supset E$, and $f^{-1}(C) = E$, i.e., $f^{-1}(C)$ is a component of $f^{-1}(D)$.

Conversely, if $f^{-1}(C)$ is supposed to be a component of $f^{-1}(D)$ and if H is a connected set such that $C \subset H \subset D$, it follows that $f^{-1}(C) \subset f^{-1}(H) \subset f^{-1}(D)$, and since the set $f^{-1}(H)$ is connected, it follows that $f^{-1}(C) = f^{-1}(H)$, which implies $C = H$ as f is onto. Hence C is a component of D. This completes the proof of the theorem.

7. Quasi-directed space

Definition 7.1. *Let (X, u) be a monotone space and let $A, B, N \in P(X)$. If A, B are semi-separated and A, N are semi-separated imply $A, B \cup N$ are semi-separated then we call (X, u) a quasi-directed monotone space (briefly, q-directed space).*

Clearly, the notion of a *q*-directed space is weaker than that of a monotone space equipped with closure preserving property, i.e., a closure space. In other words, a closure space is a *q*-directed space. But the converse is not true, in general, as shown by the following example.

Example 7.1. Let $X = \{a, b, c, d, e, f\}$. We define $u : P(X) \to P(X)$ by $u(\phi) = \phi$, $u(X) = X$, $u(\{a\}) = \{a, c, d\}$, $u(\{b\}) = \{b, c, e\}$, $u(\{e\}) = \{e\}$, $u(\{b, e\}) = \{b, c, d, e\}$ and $u(P) = X$ where $P \in P(X)$ and $P \neq \phi, \{a\}, \{b\}, \{e\}, \{b, e\}$. Let $A = \{a\}, B = \{b\}, N = \{e\}$. Clearly, each of the pair $(A, B), (A, N)$ and $(A, B \cup N)$ are semi-separated. But $u(B \cup N) \neq u(B) \cup u(N)$. Also, there is no such pair (C, D)

which are semi-separated except the above mentioned pairs, viz., $(A, B), (A, N)$ and $(A, B \cup N)$, where $C, D \in P(X)$.

We state a lemma without proof which we need in the sequel.

Lemma 7.1. *Let (X, u) be a monotone space. Let $A \subset M$ and M and N are semi-separated. Then A and N are semi-separated.*

Theorem 7.1. *Let P and C be connected sets in a q-directed space (X, u) and $P \cap C \neq \phi$. If M and N are two semi-separated sets such that $P - C = M \cup N$, then the sets $C \cup M$ and $C \cup N$ are connected.*

Proof. Let $C \cup M = A \cup B$ where A and B are semi-separated sets. So, $C \subset A \cup B$ and hence by Theorem 5.10, we assume that $C \cap A = \phi$ which gives $A \subset M$ because $A \subset C \cup M$. Now, since the sets M and N are semi-separated, the set A and N are semi-separated by Lemma 7.1; and therefore, A and $N \cup B$ are semi-separated as A and B are semi-separated and (X, u) is q-directed. Now, $P \cup C = (P - C) \cup C = M \cup N \cup C = A \cup (B \cup N)$. But $P \cup C$ is a connected set because P and C are connected sets and $P \cap C \neq \phi$. This proves the theorem.

Corollary 7.1. *If C is a connected subset of a connected q-directed space (X, u), and if M and N are two semi-separated sets such that $X - C = M \cup N$, then the sets $C \cup M$ and $C \cup N$ are connected.*

Corollary 7.2. *Let A and B be two closed sets in a q-directed space. If the sets $A \cup B$ and $A \cap B$ are connected, the sets A and B are also connected.*

Theorem 7.2. *If E is not the union of n connected sets in a monotone space (X, u), there exists $n + 1$ pairwise semi-separated sets $A_1, \cdots, A_{n+1}$ such that*

$$E = A_1 \cup \cdots \cup A_{n+1}, A_i \neq \phi \quad for \quad 1 \leq i \leq n + 1.$$

Proof. The proof is obvious.

Theorem 7.3. (Generalized Corollary 7.1) *If $C_1, \cdots, C_n$ are connected subsets of a connected q-directed space (X, u), and M and N are two semi-separated sets such that*

$$X - (C_1 \cup \cdots \cup C_n) = M \cup N,$$

then the set $C_1 \cup \cdots \cup C_n \cup M$ consists of n connected sets (which may be distinct or not).

Proof. The proof is obvious.

Corollary 7.1. gives the following results.

Theorem 7.4. *Every connected q-directed space* (X, u), *which contains more than one point, is the union of two connected sets, which are distinct from the space and contain more than one point.*

Proof. If for each x the set $X - \{x\}$ is connected, there is a decomposition $X = (X - \{x_1\}) \cup (X - \{x_2\})$, where $x_1 \neq x_2$. On the other hand, if there exists an x such that $X - \{x\}$ is not connected, it follows that $X - \{x\} = M \cup N$, where M and N are semi-separated and non-empty; therefore, by Corollary 7.1, the required decomposition of X is: $X = (M \cup \{x\}) \cup (N \cup \{x\})$. This proves the theorem.

Theorem 7.5. *Let* (X, u) *be a connected q-directed space. If E is a connected set and C is a component of* $X - E$, *then* $X - C$ *is connected.*

Proof. Assume that $X - C = M \cup N$, where M and N are semi-separated sets. Since $E \subset X - C = M \cup N$, according to Theorem 5.10, it can be assumed that $E \cap M = \phi$, which implies $E \cap (C \cup M) = \phi$ so that $C \subset C \cup M \subset X - E$. Since by Corollary 7.1, $C \cup M$ is connected, it follows by the definition of component, that $C \cup M = C$ and hence $M = \phi$. This proves the theorem.

Theorem 7.5 has the following immediate consequences.

Theorem 7.6. *If the space* (X, u) *is connected and q-directed, then every finite system S (containing at least two elements) of disjoint connected subsets contains at least two elements,* A_1 *and* B_1, *which have the following property:*

(P) There exists a connected set disjoint from A_1 *(respectively from* B_1*) which contains all the elements of S other than* A_1 *(respectively other than* B_1*).*

Proof. Let $S = (C_0, C_1, \cdots, C_n)$ and proceed by induction. Since the theorem is obvious for $n = 1$, let us assume that it holds for $n - 1$ (≥ 1).

We are going to show that there exists a number $k > 0$ such that the set C_k has the property *P*.

Suppose that the set C_1 does not possess this property. Hence there exist at least two components *A* and *B* of the set $X - C_1$ which contain the sets of the system *S*; let A be that one which does not contain C_0.

Let $m_1, \cdots, m_j$ be the sequence of indices of the sets C_i contained in *A*. It follows that

$$1 \leq j \leq n-1, \tag{7.1}$$

$$0 \neq m_1, \cdots, 0 \neq m_j, \tag{7.2}$$

$$\text{if } r \neq m_1, \cdots, r \neq m_j \text{ and } r \leq n, \text{ then } C_r \subset X - A. \tag{7.3}$$

Since the set $X - A$ is connected (by Theorem 7.5) and the system

$$S^* = (X - A, C_{m_1}, \cdots, C_{m_j})$$

contains at most *n* elements (by (7.1)), there exists by hypothesis an index $s \leq j$ such that the set C_{m_s} has the property *P* with respect to the system S^*. Therefore there exists a connected set K such that

$$(X - A) \cup C_{m_1} \cup \cdots \cup C_{m_{s-1}} \cup C_{m_{s+1}} \cup \cdots \cup C_{m_j} \subset K \subset X - C_{m_s}.$$

It follows by (7.3) that $C_q \subset K$ for every $q \neq m_s$. That means that C_{m_s} has the property *P* (with respect to the system *S*).

Finally, $m_s > 0$ by (7.2); hence m_s is the required index *k*. This completes the proof of the theorem.

Theorem 7.7. *In a connected q-directed space* (X, u), *let S be an infinite family of disjoint connected sets. If* S_0 *and* S_1 *are two arbitrary elements of S, there exists in* $X - S_0$ *or in* $X - S_1$ *a connected set, which contains infinitely many elements of S.*

Proof. Let C_j (for $j = 0, 1$) be the component of $X - S_j$ which contains S_{1-j}. Condition $S_j \subset C_{1-j} \subset X - S_{1-j}$ implies $S_{1-j} \subset X - C_{1-j} \subset X - S_j$, which implies in turn that

$$X - C_{1-j} \subset C_j \tag{7.4}$$

since the set $X - C_{1-j}$ is connected (by Theorem 7.5).

Suppose that C_0 contains only a finite number of elements of *S*. Hence there exist infinitely many elements of *S* contained in $X - C_0$ and therefore, in C_1 according to (7.4). So $X - S_1$ contains a connected set, namely C_1, which contains infinitely many elements of the family *S*. This proves the theorem.

8. Quasi-nodal sets, separating points, end points and the ending property

Definition 8.1. *A non-empty set N in a connected monotone space* (X, u) *is called quasi-nodal if the boundary of N in X is degenerate. (A set is degenerate if it is either empty or singleton).*

Definition 8.2. *Let* (X, u) *be a monotone space. A point* $p \in X$ *is said to be a separating point of a connected subset C of X provided that* $C - \{p\}$ *is not connected.*

A point $p \in X$ *is said to be an endpoint of* (X, u) *if p is not a separating point of any connected subset C of X.*

A subset E of X is said to have e.p. (the ending property) in X provided that there is no connected subset C of X such that E separates C (i.e., such that $C - E$ *is not connected).*

Remark 8.1. By definition, it follows that an endpoint of *X* has e.p.

Theorem 8.1. *The interior of any quasi-nodal set of a quasi-directed space* (X, u) *has e.p. in X.*

Proof. Let *N* be a quasi-nodal set of *X* and let C be a connected set in *X*. Let $b(N) = \{p\}$. We now consider the following two cases:

Case I. Let $p \in C$. If possible, let $C - \text{Int}\, N = A \cup B$, where *A* and *B* are non-empty, semi-separated sets. Since $p \notin \text{Int}\, N$, so let $p \in B$. Therefore, $p \notin A$. Clearly, $C = A \cup B \cup (C \cap \text{Int}\, N) = A \cup B \cup Q$, where $Q = C \cap \text{Int}\, N$. Now, $u(A) \cap Q = u(A) \cap (C \cap \text{Int}\, N) = C \cap (u(A) \cap \text{Int}\, N)$. We assert that $u(A) \cap \text{Int}\, N = \phi$. For, if $\alpha \in u(A) \cap \text{Int}\, N$ then $\alpha \in u(A)$ and $\alpha \in \text{Int}\, N$. Now $\alpha \in \text{Int}\, N$ implies *N* is a neighbourhood of α. So, by Theorem 2.6, $N \cap A \neq \phi$ as $\alpha \in u(A)$. Hence $(\text{Int}\, N \cup \{p\}) \cap A \neq \phi$, i.e., $\text{Int}\, N \cap A \neq \phi$ since $p \notin A$. This is a contradiction. Also, $A \cap u(Q) \subset A \cap u(\text{Int}\, N) \subset A \cap u(N) = A \cap (\text{Int}\, N \cup \{p\}) = (A \cap \text{Int}\, N) \cup (A \cap \{p\}) = \phi$. Hence *A* and *B* are semi-separated. Consequently, *A* and $B \cup Q$ are semi-separated as (X, u) is *q*-directed and *A*, *B* are semi-separated. So, $C = A \cup (B \cup Q)$ has a semi-separation contradicting the connectedness of *C*. Thus $C - \text{Int}\, N$ is connected.

Case II. Let $p \notin C$. The proof of this case is same as of Case I and hence omitted. This proves the theorem.

Theorem 8.2. *If E has e.p. in X, then the intersection of any connected set with* $X - E$ *is connected, where* (X, u) *is a monotone space.*

Proof. Let C be any connected set in X. So, $C \cap (X - E) = X \cap (C - E) = C - E$ is connected since E has e.p. in X.

Theorem 8.3. *Let* (X, u) *be a quasi-directed space. Then the components of any set E that has e.p. in X also have e.p. in X.*

Proof. Let P be any connected set in X and C be any component of E. If $P \cap C = \phi$, then clearly, $P - C = P$ is connected. Now, let $P \cap C \neq \phi$. If possible, let $P - C = M \cup N$ where $u(M) \cap N = \phi = M \cap u(N)$. Now, $P - E \subset P - C$ $= M \cup N$. So, by Theorem 5.10, it can be assumed that $(P - E) \cap M = \phi$ as $P - E$ is connected since E has e.p. in X. Hence $(P - E) \cap (C \cup M) = \phi$, because $P - E \subset P - C$. Consequently, $C \cup M \subset E$. For, if $C \cup M \not\subset E$ then there exists a $p \in C \cup M$ such that $p \notin E$. Clearly, $p \notin C$ as $C \subset E$. So, $p \in M$ and $p \notin E$. Therefore, $p \in P - E$ as $M \subset P$. Thus $p \in (P - E) \cap (C \cup M)$ which contradicts that $(P - E) \cap (C \cup M) = \phi$. Now, by Theorem 7.1, $C \cup M$ is a connected set contained in E. But C is a component of E, so, $C \subset C \cup M$ implies $M = \phi$. So, $P - C$ is connected which implies that C has e.p. in X. This proves the theorem.

9. Endings and boundaries of endings

Definition 9.1. *A non-empty connected set with e.p. in a monotone space* (X, u) *is called an ending of X.*

Theorem 6.1, Theorem 6.2 (ii) and Theorem 8.3 enable us to disassemble a set with e.p. into endings in a *q*-directed space. Reassembly raises some difficulties. However, we have the following.

Theorem 9.1. *The union of a finite collection of sets with e.p. in a monotone space* (X, u) *has e.p. in X.*

Proof. Let E and F have e.p. in X and let C be any connected set in X. Now, since E has e.p. in X, $C - E$ is connected. Again, since F has e.p. in X, so, $(C - E) - F$ is connected. But $C - (E \cup F) = (C - E) - F$. So, $C - (E \cup F)$ is connected. Hence $E \cup F$ has e.p. in X. This proves the theorem.

Obviously, the intersection of two sets with e.p. in X need not have e.p. in X. However, the intersection of a nested collection is more tractable.

Theorem 9.2. *If* $\mathcal{E}$ *is a nested collection of sets with e.p. in a monotone space* (X, u), *then* $\cap\mathcal{E}$ *has e.p. (possibly trivially) in X.*

Proof. Let $N = \cap\mathcal{E}$ and *suppose* there is a connected set C such that $C - N$ is the union of semi-separated sets A and B. Since A can not be contained in all the elements of $\mathcal{E}$, there is a member E_A of $\mathcal{E}$ such that $A - E_A \neq \phi$. Similarly, there is a set E_B of $\mathcal{E}$ such that $B - E_B \neq \phi$. Let $E = E_A \cap E_B$. Since $\mathcal{E}$ is nested, $E \in \mathcal{E}$ and $A - E$ and $B - E$ are both non-empty. But then $C - E = (A - E) \cup (B - E)$, a semi-separation. This is a contradiction. Hence $C - N$ is connected. This proves the theorem.

Theorem 9.3. *If* p *is any point of a connected monotone space* (X, u), *then there is an ending of* X *that is minimal with respect to being an ending of* X *and containing p.*

Proof. X is (trivially) an ending of X that contains p. Form a maximal nest $\mathcal{E}$ of endings of X that contain p, and let $E = \cap\mathcal{E}$.

Definition 9.2. *A boundary point p of a set E is called an external boundary point of* E *if* $p \notin E$.

Theorem 9.4. *Let E be an ending of a monotone space* (X, u) *and let B be the set of external boundary points of E. Then B cannot contain two mutually semi-separated sets.*

Proof. Let A and C be two mutually semi-separated sets such that $A \cup C \subset B$. Now, $E \subset E \cup A \cup C \subset E \cup B = u(E)$. So, by Corollary 5.3, $E \cup A \cup C$ is connected. But $(E \cup A \cup C) - E = A \cup C$, a semi-separation as $E \cap (A \cup C) = \phi$. This is a contradiction since E has e.p. in X. This proves the theorem.

Corollary 9.1. *The external boundary of any ending of a monotone space* (X, u) *is connected.*

Corollary 9.2. *For any two points of external boundary B of any ending of a monotone space X, one is a cluster point of the other. Hence, if X is* T_1, *then B is degenerate.*

Proof of Corollary 9.2. Let α, $\beta \in B$. If possible, let $u(\{\alpha\}) \cap \{\beta\} = \phi$ and $u(\{\beta\}) \cap \{\alpha\} = \phi$. Then $\{\alpha\}$ and $\{\beta\}$ are two mutually semi-separated sets contained in B, a contradiction of Theorem 9.4. Hence for any two points of B of any ending of X, at least one is a cluster point of the other. Hence, if X is T_1, then B is degenerate.

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