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Normal Functions and Normal Families

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Abstract. In this paper, we prove the following theorem: Let F be a family of holomorphic functions in the unit disc D and let a be a nonzero complex number. If, for any $f \in F$, $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, then F is uniformly normal in D, that is, there exists a positive constant M such that $(1 - |z|^2) f^{\#}(z) \le M$ for each $f \in F$ and $z \in D$, where M is independently of *f*. This result improves related results due to [2], [8], and [3].

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1. Introduction

Let f and g be two meromorphic functions, and let a be a complex number. If $g(z) = a$ whenever $f(z) = a$, we denote it by $f = a \Rightarrow g = a$, and $f = a \Leftrightarrow g = a$ means $f(z) = a$ if and only if $g(z) = a$.

Let *D* denote the unit disk in the complex plane ***C***. A function *f* meromorphic in *D* is called a normal function, in the sense of [6], if there exist a constant $M(f)$ such that $(1 - |z|^2) f^{\#}(z) \le M(f)$ for each $z \in D$, where

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative.

Let *F* be a family of meromorphic functions defined in *D*. *F* is said to be normal in *D* (see [9]), in the sense of Montel, if for any sequence $f_n \in F$ there exists a subsequence f_{n_j}, such that f_{n_j} converges spherically, locally and uniformly in *D*, to a meromorphic function or ∞.

Suppose that F is a family of functions meromorphic in D such that each function of F is a normal function, then, for each function $f \in F$, there exists a constant $M(f)$ such that

$$\left(1-|z|^2\right) f^{\#}(z) \leq M(f)$$

for each $z \in D$. In general, $M(f)$ is a constant dependent on f, and we can not conclude that $\{M(f), f \in F\}$ is bounded. If $\{M(f), f \in F\}$ is bounded, we give the definition as follows

Definition. *Let F be a family of meromorphic functions in the unit disc D. If there exists a positive constant M such that*

$$\sup\left\{\left(1-|z|^2\right) f^{\#}(z) : z \in D,\ f \in F\right\} < M,$$

we call the family F a uniformly normal family in D.

Remark 1. The idea of this definition is suggested by Pang (see [7]).

Remark 2. A well-known result due to Marty (see [4], [9] and [11]) says that a family F of functions meromorphic in D is a normal family if and only if for each compact subset K of D there exists a constant M_K such that $f^{\#}(z) \leq M_K$ for each $f \in F$ and for each $z \in K$. Clearly, by Marty's criterion if F is a uniformly normal family in D, then F must be normal in D. However, it is obvious that the converse is not always true.

It is natural to ask: *When is a normal family F in D also uniformly normal in D*? (The question is first introduced by Bergweiler and Pang (see [7]).)

Schwick [10] discovered a connection between normality criteria and sharing values. He proved

Theorem A. *Let F be a family of meromorphic functions in the unit disc D and let* a_1, a_2 *and* a_3 *be distinct complex numbers. If, for any* $f \in F$, $f(z) = a_i \Leftrightarrow f'(z) = a_i\ (i = 1, 2, 3)$, *then F is normal in D.*

Pang [7] proved that the family F in Theorem A is also uniformly normal, as follows

Theorem B. *Let F be a family of meromorphic functions in the unit disc D and let* a_1, a_2 *and* a_3 *be distinct complex numbers. If, for any* $f \in F$,

$f(z) = a_i \Leftrightarrow f'(z) = a_i \; (i = 1, 2, 3)$, *then F is uniformly normal in D, that is, there exists a positive constant M such that*

$$\left(1 - |z|^2\right) f^{\#}(z) \leq M$$

for each $f \in F$ *and* $z \in D$, *where M is independent of f.*

Remark 3. In fact, from the proof in [7], we see that Theorem B still remains true if $f(z) = a_i \Rightarrow f'(z) = a_i \; (i = 1, 2, 3)$ for any $f \in F$.

Chen and Hua [2], Pang and Zalcman [8] proved the following normality criterion.

Theorem C. *Let F be a family of holomorphic functions in the unit disc D and let a be a nonzero complex number. If, for any* $f \in F$, $f(z) = a \Leftrightarrow f' = a$, $f' = a \Rightarrow f''(z) = a$, *then F is normal in D.*

In [3], Fang improved Theorem C as follows

Theorem D. *Let F be a family of holomorphic functions in the unit disc D and let a be a nonzero complex number. If, for any* $f \in F$, $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, *then F is normal in D.*

In this paper, by using a method different from that used in [3], we obtain the following stronger result.

Theorem 1. *Let F be a family of holomorphic functions in the unit disc D and let a be a nonzero complex number. If, for any* $f \in F$, $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, *then F is uniformly normal in D, that is, there exists a positive constant M such that*

$$\left(1 - |z|^2\right) f^{\#}(z) \leq M$$

for each $f \in F$ *and* $z \in D$, *where M is independent of f.*

Remark 4. The following example (see [2] and [3]) shows that $a \neq 0$ cannot be omitted in Theorem 1.

Let $F = \{ f_n(z) = e^{nz} : n = 1, 2, 3 \cdots \}$, $D = \{ z : |z| < 1 \}$. Then, for every $f_n \in F$, it is easy to see that $f_n(z) = 0 \Rightarrow f_n'(z) = 0 \Rightarrow f_n''(z) = 0$. However, $f_n^{\#}(0) = n/2 \to \infty$ *as* $n \to \infty$, thus *F* is not uniformly normal in *D*.

We shall use the standard notations in Nevanlinna theory (see [4], [11]).

2. Lemmas

For convenience, we define

$$LD(r, f) := c_1 m\left(r, \frac{f'}{f}\right) + c_2 m\left(r, \frac{f''}{f'}\right) + c_3 m\left(r, \frac{f'}{f-a}\right)$$
$$+ c_4 m\left(r, \frac{f''}{f-a}\right) + c_5 m\left(r, \frac{f''}{f'-a}\right), \quad (a \in \mathbf{C})$$

where c_1, c_2, c_3, c_4, c_5 are constants, which may have different values at different occurrences.

Lemma 1. *Let f be a non-constant holomorphic functions on the unit disc D, and a be a nonzero complex number. Let*

$$\psi(z) := \psi(f(z)) = \frac{f'(z) + f''(z)}{f(z) - a} - \frac{2f''(z)}{f'(z) - a}.$$

If $f = a \Rightarrow f' = a$, $f' = a \Rightarrow f'' = a$ *on D, and* $f(0) \neq a$, $f'(0) \neq a$, $f''(0) \neq 0$, $f'(0) \neq f''(0)$ *and* $\psi(0) \neq 0$, *then*

$$T(r, f) \leq LD(r, f) + O(1) + 3\log\frac{|f(0) - a|}{|f''(0) - f'(0)|}$$
$$+ \log\frac{|(f(0) - a)(f'(0) - a)|}{|f''(0)|} + 2\log\frac{1}{|\psi(0)|}.$$

Proof. Let $f(z_0) = a$. By the assumptions we may assume that, near z_0

$$f(z) = a + a(z - z_0) + \frac{a}{2}(z - z_0)^2 + b(z - z_0)^3 + O\left((z - z_0)^4\right),$$

where $b = f^{(3)}(z_0)/6$ is a constant. Then

$$f'(z) = a + a(z - z_0) + 3b(z - z_0)^2 + O\left((z - z_0)^3\right),$$

$$f''(z) = a + 6b(z - z_0) + O\left((z - z_0)^2\right),$$

and thus

$$\frac{f'(z)+f''(z)}{f(z)-a}=\frac{2}{z-z_0}+\frac{6b}{a}+O(z-z_0),$$

$$\frac{2f''(z)}{f'(z)-a}=\frac{2}{z-z_0}+\frac{6b}{a}+O(z-z_0).$$

Hence $\psi(z_0)=0$, and

$$\begin{aligned} N\left(r,\frac{1}{f-a}\right) &\le N\left(r,\frac{1}{\psi}\right)\le T(r,\psi)+\log\frac{1}{|\psi(0)|} \\ &\le N(r,\psi)+LD(r,f)+\log\frac{1}{|\psi(0)|} \qquad (2.1)\\ &= N_0\left(r,\frac{1}{f'-a}\right)+LD(r,f)+\log\frac{1}{|\psi(0)|}, \end{aligned}$$

where $N_0(r,1/(f'-a))$ is the counting function for the zeros of $f'-a$ which are not zeros of $f-a$. Since $f=a\Rightarrow f'=a$, form (2.1) we get

$$2N\left(r,\frac{1}{f-a}\right)\le N\left(r,\frac{1}{f'-a}\right)+LD(r,f)+\log\frac{1}{|\psi(0)|}. \qquad (2.2)$$

On the other hand, by the assumptions we have

$$\begin{aligned} N\left(r,\frac{1}{f'-a}\right) &\le N\left(r,\frac{1}{\frac{f''}{f'}-1}\right)\le T\left(r,\frac{f''}{f'}\right)+\log\frac{|f'(0)|}{|f''(0)-f'(0)|}+O(1) \\ &= N\left(r,\frac{f''}{f'}\right)+\log\frac{|f'(0)|}{|f''(0)-f'(0)|}+LD(r,f)+O(1) \qquad (2.3)\\ &= \overline{N}\left(r,\frac{1}{f'}\right)+\log\frac{|f'(0)|}{|f''(0)-f'(0)|}+LD(r,f)+O(1). \end{aligned}$$

Next we need the estimate of $\overline{N}(r, \frac{1}{f'})$. Since

$$
\begin{aligned}
m\left(r, \frac{1}{f-a}\right) &\le m\left(r, \frac{1}{f'}\right) + LD(r, f) \\
&\le T(r, f') - N\left(r, \frac{1}{f'}\right) + LD(r, f) + \log\frac{1}{|f'(0)|} \\
&\le T(r, f) - N\left(r, \frac{1}{f'}\right) + LD(r, f) + \log\frac{1}{|f'(0)|} \\
&\le T\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + LD(r, f) + O(1) + \log\frac{|f(0)-a|}{|f'(0)|} \\
&= m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + LD(r, f) + O(1) \\
&\quad + \log\frac{|f(0)-a|}{|f'(0)|},
\end{aligned}
$$

we obtain

$$
N\left(r, \frac{1}{f'}\right) \le N\left(r, \frac{1}{f-a}\right) + LD(r, f) + O(1) + \log\frac{|f(0)-a|}{|f'(0)|}. \tag{2.4}
$$

Thus, from (2.2), (2.3) and (2.4) , we get

$$
N\left(r, \frac{1}{f-a}\right) \le LD(r, f) + O(1) + \log\frac{|f(0)-a|}{|f''(0)-f'(0)|} + \log\frac{1}{|\psi(0)|}, \tag{2.5}
$$

$$
N\left(r, \frac{1}{f'-a}\right) \le LD(r, f) + O(1) + 2\log\frac{|f(0)-a|}{|f''(0)-f'(0)|} + \log\frac{1}{|\psi(0)|}. \tag{2.6}
$$

Using Milloux's inequality, we have

$$
\begin{aligned}
T(r, f) \le N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'-a}\right) + LD(r, f) + O(1) \\
+ \log\frac{|(f(0)-a)(f'(0)-a)|}{|f''(0)|}.
\end{aligned}
$$

Substituting (2.5) and (2.6) in the above inequality yields the conclusion.

Lemma 2. (Bureau [1]) *Let* b_1, b_2, and b_3 *be positive numbers and* $U(r)$ *a nonnegative, increasing and continuous function on an interval* $[r_0, R)$, $R < \infty$. *If*

$$U(r) \le b_1 + b_2 \log^+ \frac{1}{\rho - r} + b_3 \log^+ U(\rho)$$

for any $r_0 < r < \rho < R$, *then*

$$U(r) \le B_1 + B_2 \log^+ \frac{1}{R - r}$$

for $r_0 \le r < R$, *where* B_1 *and* B_2 *depend on* $b_i (1 = 1, 2, 3)$ *only*.

Lemma 3. (see Hiong [4]) *If* $f(z)$ *is meromorphic in a disk* $|z| < R$ *such that* $f(0) \ne 0, \infty$, *then, for* $o < r < \rho < R$,

$$m\left(r, \frac{f^{(k)}}{f}\right) \le C_k \left\{1 + \log^+ \log^+ \frac{1}{|f(0)|} + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho - r} + \log^+ \rho + \log^+ T(\rho, f)\right\},$$

where C_k *is a constant depending only on* k.

The following is the wellknown Zalcman's lemma [12].

Lemma 4. *Let F be a family of functions meromorphic in a domain D. If F is not normal at* $z_0 \in D$, *then there exist a sequence of points* $z_n \in D$, $z_n \to z_0$, *a sequence of positive numbers* $\rho_n \to 0$, *and a sequence of functions* $f_n \in F$ *such that*

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function on **C**.

3. Proof of Theorem 1

Proof. Suppose that *F* is not uniformly normal in *D*. Then, we can find $f_n \in F$, $z_n \in D$, such that

$$g_n(z) = f_n\left(z_n + (1-|z_n|^2)\,z\right)$$

satisfies

$$g_n^{\#}(0) = \left(1-|z_n|^2\right) f_n^{\#}(z_n) \to \infty$$

as $n \to \infty$. It follows that $\{g_n(z)\}$ is not normal at $z = 0$. We distinguish two cases:

(1) $g_n = g_n'$ for every $n \in \boldsymbol{N}$ then $g_n(z) = C_n e^z$, which is normal at $z = 0$.

(2) Consider the case that g_n and g_n' are not identical. By Lemma 4, there exist a subsequence g_n (without loss generality, we may assume g_n), a sequence $\eta_n \in D, \eta_n \to 0$, and a positive sequence $\rho_n \to 0$ such that

$$G_n(\zeta) = g_n(\eta_n + \rho_n\zeta) = f_n\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\zeta\right)$$

converges uniformly to a non-constant entire function $G(\zeta)$ on each compact subset of $\boldsymbol{C}$. Thus, for any positive integer *k*,

$$G_n^{(k)}(\zeta) = \left(1-|z_n|^2\right)^k \rho_n^k f_n^{(k)}\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\zeta\right). \tag{3.1}$$

We claim that $G(\zeta)$ is not a polynomial of degree less than 3. Indeed, if $G(\zeta)$ is a polynomial, then there exists a point ζ_0 such that $G(\zeta_0) = a$. By Hurwitz' theorem, there is a sequence $\zeta_n \to \zeta_0$ such that

$$G_n(\zeta_n) = g_n(\eta_n + \rho_n\zeta_n) = f_n\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\zeta_n\right) = a$$

for *n* sufficiently large. It follows from the hypotheses on F that

$$f_n'\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\,\zeta_n\right)$$
$$= f_n''\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\,\zeta_n\right) = a$$

for n sufficiently large. On the other hand, by (3.1), we have

$$G_n'(\zeta_n) = \left(1-|z_n|^2\right)\rho_n f_n'\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\zeta_n\right) \to G'(\zeta_0),$$

$$G_n''(\zeta_n) = \left(1-|z_n|^2\right)^2 \rho_n^2 f_n''\left(z_n + (1-|z_n|^2)\,\eta_n + (1-|z_n|^2)\,\rho_n\zeta_n\right) \to G''(\zeta_0).$$

Thus $G'(\zeta_0) = G''(\zeta_0) = 0$.

Choose ζ_1 with

$$G(\zeta_1) \neq 0, a\,;\; G'(\zeta_1) \neq 0\,; G''(\zeta_1) \neq 0. \tag{3.2}$$

Then

$$\frac{1}{\rho_n^2 \left(1-|z_n|^2\right)^2} \times \frac{f_n\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) - a}{f_n''\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) - f_n'\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right)} \to \frac{G(\zeta_1)}{G''(\zeta_1)},$$

$$\frac{1}{\rho_n\left(1-|z_n|^2\right)} \times \frac{\left(f_n\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) - a\right)\left(f_n'\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) - a\right)}{f_n''\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right)} \to \frac{G'(\zeta_1)(G(\zeta_1)-a)}{G''(\zeta_1)}.$$

On the other hand, we claim that there are only finitely many f_n such that $\psi(f_n) \equiv 0$. Indeed, suppose that there is a subsequence $\{f_{n_j}\} \subset \{f_n\}$ such that $\psi(f_{n_j}) \equiv 0$.

Then

$$\frac{f_{n_j}'\left(z_{n_j} + (1-|z_{n_j}|^2)\eta_{n_j} + (1-|z_{nj}|)^2\rho_{n_j}\zeta\right) + f_{n_j}''\left(z_{n_j} + (1-|z_{n_j}|^2)\eta_{n_j} + (1-|z_{n_j}|^2)\rho_{n_j}\zeta\right)}{f_{n_j}\left(z_{n_j} + (1-|z_{n_j}|^2)\eta_{n_j} + (1-|z_{n_j}|^2)\rho_{n_j}\zeta\right) - a}$$

$$= \frac{2f_{n_j}''\left(z_{n_j} + (1-|z_{n_j}|^2)\eta_{n_j} + (1-|z_{n_j}|^2)\rho_{n_j}\zeta\right)}{f_{n_j}'\left(z_{n_j} + (1-|z_{n_j}|^2)\eta_{n_j} + (1-|z_{n_j}|^2)\rho_{n_j}\zeta\right) - a},$$

and thus

$$\frac{\rho_{n_j}\left(1-\left|z_{n_j}\right|^2\right)G'_{n_j}(\zeta)+G''_{n_j}(\zeta)}{G_{n_j}(\zeta)-a}=\frac{2\rho_{n_j}\left(1-\left|z_{n_j}\right|^2\right)G''_{n_j}(\zeta)}{G'_{n_j}(\zeta)-a\rho_{n_j}\left(1-\left|z_{n_j}\right|^2\right)}.$$

Letting $j\to\infty$, we get $G''(\zeta)\equiv 0$, a contradiction. Then we may assume that $\psi(f_n)\not\equiv 0$, for all n. Thus

$$\rho_n^2\psi_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)\to\frac{G''(\zeta)}{G(\zeta_1)-a}, \tag{3.3}$$

where $\psi_n=\psi(f_n)$. So we have

$$\log\frac{\left|f_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)-a\right|}{\left|f''_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)-f'_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)\right|}\to-\infty, \tag{3.4}$$

$$\log\frac{\left|\left(f_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)-a\right)\left(f'_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)-a\right)\right|}{\left|f''_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)\right|}\to-\infty, \tag{3.5}$$

and

$$\log\frac{1}{\left|\psi_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\right)\right|}\to-\infty, \tag{3.6}$$

as $n\to\infty$. For $n=1,2,\cdots$, set

$$P_n(z)=f_n\left(z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1+z\right).$$

Let n be sufficiently large. Then P_n is defined and holomorphic on the disk $0<|z|<\frac{1}{2}$, since

$$z_n+(1-|z_n|^2)\eta_n+(1-|z_n|^2)\rho_n\zeta_1\to 0.$$

By (3.2) and (3.3), we have

$$P_n(0) = G_n(\zeta_1) \to G(\zeta_1) \neq 0,\ a, \tag{3.7}$$

$$P_n'(0) = \frac{1}{(1-|z_n|^2)\ \rho_n} G_n'(\zeta_1) \to \infty, \tag{3.8}$$

$$P_n''(0) = \frac{1}{(1-|z_n|^2)^2 \rho_n^2} G_n''(\zeta_1) \to \infty, \tag{3.9}$$

$$\psi(P_n(0)) = \psi_n\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) \to \infty. \tag{3.10}$$

Therefore, by (3.7)–(3.10) we may apply Lemma 1 to $P_n(z)$, and using (3.4), (3.5) and (3.6) we obtain

$$T(r, P_n) \leq LD(r, P_n),$$

for sufficiently large n. Hence by Lemma 2 and Lemma 3, we get

$$T\left(\frac{1}{4}, P_n\right) \leq M,$$

where M is a constant independent of n. It follows that $f_n(z)$ are bounded for sufficiently large n and $|z| < \frac{1}{8}$. But, from

$$\left(1-|z_n|^2\right)^2 \rho_n^2 f_n''\left(z_n + (1-|z_n|^2)\eta_n + (1-|z_n|^2)\rho_n\zeta_1\right) = G_n''(\zeta_1) \to G''(\zeta_1) \neq 0$$

we know that $f_n(z)$ cannot be bounded in $|z| < \frac{1}{8}$. We arrive at a contradiction. This completes the proof of Theorem 1.

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