BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY

*Bull. Malays. Math. Sci. Soc.* (2) **27** (2004), 161–168

# A Framed $f(3,-1)$ Structure on the Cotangent Bundle of a Hamilton Space

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**Abstract.** For the cotangent bundle $(T^*M, \tau^*, M)$ of a smooth manifold *M*, the kernel of a differential $\tau^*_*$ of the projection $\tau^*$ defines the vertical subbundle $VT^*M$ of the bundle $(TT^*M, \tau_{T^*M}, T^*M)$. A supplement $HT^*M$ of it is called a horizontal subbundle or a nonlinear connection on *M*, [6,7]. The direct decomposition $TT^*M = HT^*M \oplus VT^*M$ gives rise to a natural almost product structure *P* on the manifold $T^*M$. A general method to associate to *P* a framed $f(3,-1)$ - structure of any corank is pointed out. This is similar to that given by us in [2] for the tangent bundle of a Lagrange space. When we endow *M* with a regular Hamiltonian *H* and use as the nonlinear connection that canonically induced by *H*, a framed $f(3,-1)$ - structure $P_2$ of corank 2 naturally appears on $T^*M$. This reduces to that found by us in [3] when $H = K^2$, for *K* the fundamental function of a Cartan space $K^n = (M, K)$. Then we show that on some conditions for *H* the restriction of $P_2$ to the submanifold $H = 1$ of $T_0^*M$ provides an almost paracontact structure on this submanifold. The conditions taken on H hold for the $\varphi$ - Hamiltonians introduced by us in [4] as well as for $H = K^2$. The idea of this study has the origin in the paper [1] of M. Anastasiei.

2000 Mathematics Subject Classification: 53C60

## 1. A framed $f(3,-1)$ structure on $T^*M$

Let $M$ be a smooth i.e. $C^\infty$ manifold of dimension $n$ with local coordinates $(x^i)$, $i, j, k, \cdots = 1, \cdots, n$. And let $(T^*M, \tau^*, M)$ be its cotangent bundle. On $T^*M$ we shall take as local coordinates $(x^i \equiv x^i \circ \tau, p_i)$, where $(p_i)$ are the coordinates of a covector from $T_x^*M$, $x(x^i)$, in the natural cobasis $(dx^i)$.

The set $VT^*M = \bigcup_{u \in T^*M} V_u T^*M$ for $V_u T^*M = \ker \tau^*_{*,u}$, projected over $T^*M$ gives the vertical bundle over $T^*M$. A supplement $HT^*M$ of it is called horizontal bundle or a nonlinear connection on *M*. We have the decomposition

$$T_u T^*M = H_u T^*M \oplus V_u T^*M \,,\; u \in T^*M \,. \tag{1.1}$$

The distribution $u \to V_u T^*M$ is locally spanned by $\dot{\partial}^i := \frac{\partial}{\partial p_i}$ and one takes $\delta_i = \partial_i + N_{ik}(x,p)\dot{\partial}^k$ as a local basis for the horizontal distribution $u \to H_u T^*M$. Thus the basis $(\delta_i, \dot{\partial}^i)$ is adapted to the decomposition (1.1). The Einstein convention on summation over the indices $i, j, k, \cdots$ is implied.

The linear operator $P$ on $T_u T^*M$ defined by

$$P(\delta_i) = \delta_i \,,\; P(\dot{\partial}^i) = -\dot{\partial}^i \,, \tag{1.2}$$

gives an almost product on $T^*M$, that is $P^2 = I$, where I is the identity operator.

The dual basis of $(\delta_i, \dot{\partial}^i)$ is $(dx^i, \delta p_i = dp_i - N_{ij}(x,p)\,dx^j)$.

Let $\xi_1, \xi_2, \cdots, \xi_r$ be $r$ linearly independent horizontal vector fields and $\zeta_1, \zeta_2, \cdots, \zeta_s$ be $s$ linearly independent vertical vector fields on $T^*M$, such that $m = r + s < 2n$. We consider also the $r$ horizontal 1-forms $\omega_1, \omega_2, \cdots, \omega_r$ ($\omega_\alpha = \omega_{\alpha i} dx^i$, $\alpha, \beta, \cdots = 1, \cdots, r$) and $s$ vertical 1-forms $\eta_1, \eta_2, \cdots, \eta_s$ ($\eta_a = \eta_a^i \delta p_i$, $a, b, \cdots = 1, \cdots, s$) such that

$$\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta} \,,\; \eta_a(\zeta_b) = \delta_{ab} \,. \tag{1.3}$$

Notice that we have also

$$\omega_\alpha(\zeta_a) = 0 \,,\; \eta_a(\xi_\alpha) = 0 \,. \tag{1.3$'$}$$

We clearly have $P(\xi_\alpha) = \xi_\alpha$, $P(\zeta_a) = -\zeta_a$, $\forall\, \alpha, a$ and

**Lemma 1.1.** $\omega_\alpha \circ P = \omega_\alpha, \;\; \eta_a \circ P = -\eta_a, \;\; \forall\, \alpha, a$.

Now we put

$$P_m = P - \sum_\alpha \omega_\alpha \otimes \xi_\alpha + \sum_a \eta_a \otimes \zeta_a \tag{1.4}$$

and we have

**Theorem 1.1.** *The triple* $\mathsf{F}_m = (P_m, (\xi_\alpha, \zeta_a), (\omega_\alpha, \eta_a))$ *defines a framed* $f(3,-1)$*-structure on* $T^*M$, *that is, we have*

$$P_m(\xi_\alpha) = 0, P_m(\zeta_a) = 0, \omega_\alpha \circ P_m = 0,\ \eta_a \circ P_m = 0,\ \forall \alpha, a$$
$$P_m^2 = I - \sum_\alpha \omega_\alpha \otimes \xi_\alpha - \sum_a \eta_a \otimes \zeta_a . \tag{1.5}$$

*Proof.* One uses (1.3), (1.3)' and the Lemma 1.1.

This result is completed by

**Theorem 1.2.** *The operator $P_m$ is of rank $2n - m$ and it satisfies*

$$P_m^3 - P_m = 0 . \tag{1.6}$$

*Proof.* The equality (1.6) follows from (1.5). In order to prove that rank $P_m = 2n - m$, we show that $\ker P_m$ is spanned by the vector fields $(\xi_\alpha, \zeta_a)$, $\alpha = 1,\cdots,r$, $a = 1,\cdots,s$, $r + s = m$. By (1.5), Span$(\xi_\alpha, \zeta_a)$ is contained in $\ker P_m$. For proving the converse inclusion, let be $Z = X^i\delta_i + Y_i\dot{\partial}^i \in \ker P_m$. Then by (1.4),

$$P_m(Z) = X^i\delta_i - Y_i\dot{\partial}^i - \sum_\alpha \left(\omega_{\alpha k}X^k\right)\xi^i_\alpha\delta_i + \sum_a \left(\eta^k_a Y_k\right)\zeta_{ia}\dot{\partial}^i \quad \text{and } P_m(Z) = 0$$

gives
$$X^i = \sum_\alpha \left(\omega_{\alpha k}X^k\right)\xi^i_\alpha ,\ Y_i = \sum_a \left(\eta^k_a Y_k\right)\zeta_{ia} .$$

It follows

$$Z = \sum_\alpha \left(\omega_{\alpha k}X^k\right)\xi_\alpha + \sum_a \left(\eta^k_a Y_k\right)\zeta_a ,\ \text{hence } Z \in \text{Span}\left(\xi_\alpha, \zeta_a\right).$$

Theorem 1.2 says that the framed $f(3,-1)$-structure $\mathsf{F}_m$ is of corank *m*. The term $f(3,-1)$-structure is suggested by (1.6). We refer to the book [5] for an account of framed $f(3,-1)$-structures and the other related structures.

The existence of $\mathsf{F}_m$ is heavily based on the existence of linearly independent vector fields $\xi_\alpha, \zeta_a$.

In the next section we shall exhibit a natural framed $f(3,-1)$-structure on $T^*M$ when *M* is a Hamilton space.

## 2. A framed $f(3,-1)$-structure on $T^*M$, when *M* is a Hamilton space

A Hamilton space is a pair $(M, H)$, where $H : T^*M \to \mathrm{R}$ is a smooth regular Hamiltonian. This means that the matrix with the entries

$$g^{ij}(x,y) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j\ H(x,p) \tag{2.1}$$

is of rank $n$.

The regular Hamiltonian $H$ induces (see Ch. 4 in [7]) a nonlinear connection of local coefficients

$$N_{ij}(x,p) = \frac{1}{4}\left\{g_{ij}, H\right\} - \frac{1}{4}\left(g_{ik}\dot{\partial}^k\partial_j\ H + g_{jk}\ \dot{\partial}^k\partial_i H\right), \tag{2.2}$$

where $\{,\}$ denotes the usual Poisson brackets and $g_{ij}$ denotes the inverse of the matrix $(g^{jk})$. Thus we may consider the almost product structure $P$ completely determined by $H$.

Assume that $g^{ij}(x,p)p_i p_j > 0$ on the slit cotangent bundle $T_0^*M = T^*M \setminus 0$ and set $\varepsilon^2 = g^{ij}(x,p)\,p_i p_j$. From now on we restrict our considerations to $T_0^*M$.

We consider the vector fields

$$\xi = \frac{1}{\varepsilon}p^i\delta_i,\ \ \zeta = \frac{1}{\varepsilon}p_i\dot{\partial}^i \tag{2.3}$$

and the 1-forms

$$\omega = \frac{1}{\varepsilon}\left(g_{ij}p^j\right)dx^i,\ \ \eta = \frac{1}{\varepsilon}\left(g^{ij}p_j\right)\delta p_i\,. \tag{2.3)'$$

It follows that

$$\omega(\xi) = 1,\ \ \eta(\zeta) = 1 \tag{2.4}$$

$$\omega(\zeta) = 0,\ \ \eta(\xi) = 0\,, \tag{2.4)'$$

and the Lemma 1 holds for $\alpha = a = 1,\ \xi_1 = \xi,\ \zeta_1 = \zeta,\ \omega_1 = \omega,\ \eta_1 = \eta$.

We set

$$P_2 = P - \omega\otimes\xi + \eta\otimes\zeta\,. \tag{2.5}$$

Using (2.3), (2.3)' and Lemma 1 for the present case, one gets

**Theorem 2.2.** *The triple* $\mathsf{F}_2 = (P_2, (\xi,\zeta), (\omega,\eta))$ *is a framed* $f(3,-1)$*-structure on* $T_0^*M$, *that is,*

$$P_2(\xi) = P_2(\zeta) = 0, \quad \omega \circ P_2 = \eta \circ P_2 = 0,$$
$$P_2^2 = I - \omega \otimes \xi - \eta \otimes \zeta. \tag{2.6}$$

**Remark 2.1.** The framed $f(3,-1)$-structure $\mathsf{F}_2$ is of corank 2 and depends only on the Hamiltonian $H$ on $T_0^* M$.

We consider $T_0^*M$ as a Riemannian manifold with the Sasaki type metric

$$G(x, p) = g_{ij} dx^i \otimes dx^j + g^{ij} \delta p_i \otimes \delta p_j. \tag{2.7}$$

One easily checks that

$$\omega(X) = G(X, \xi), \quad \eta(X) = G(X, \zeta), \quad \forall\ X \in \chi(T_0^*M). \tag{2.8}$$

We have

**Theorem 2.3.** *The Riemannian metric G satisfies*

$$G(P_2X, P_2Y) = G(X, Y) - \omega(X)\omega(Y) - \eta(X)\eta(Y), \quad \forall\ X, Y \in \chi\big(T_0^*M\big). \tag{2.9}$$

*Proof.* First, we notice that $G(\xi, \xi) = G(\zeta, \zeta) = 1$ and $G(\xi, \zeta) = 0$ and we have that $G(PX, \xi) = \omega(PX) = \omega(X)$, $G(PX, \zeta) = -\eta(X)$ by (2.8) and Lemma 1.1. Then we have

$$G\big(PX - \omega(X)\xi + \eta(X)\zeta, PY - \omega(Y)\xi + \eta(Y)\zeta\big) = G\big(PX, PY\big) - \omega(Y)G(PX, \xi)$$
$$+ \eta(Y)G(PX, \xi) - \omega(X)G(PY, \xi) + \omega(X)\omega(Y) + \eta(X)G(PY, \zeta) + \eta(X)\eta(Y)$$
$$= G(X, Y) - \omega(X)\omega(Y) - \eta(X)\eta(Y),$$

because of $G(PX, PY) = G(X, Y)$.

Theorem 2.3 says us that $(\mathsf{F}_2, G)$ is a Riemannian framed $f(3,-1)$-structure on $T_0^* M$.

## 3. On structure induced by $F_2$ on the indicatrix bundle over $T_0^*M$

The set $I_H = \{(x, p) \in T_0^*M \,|\, H(x, p) = 1\}$ is a $(2n-1)$-dimensional submanifold on $T_0^*M$. We call it the indicatrix bundle of the Hamilton space $H^n = (M, H)$, extending a term used in Finsler geometry.

We consider again $T_0^*M$ as a Riemannian manifold with the Sasaki type metric $G$.

We are interested to find the unit normal vector field to $I_H$. We recall that $G(\xi,\xi) = 1$ and $G(\zeta,\zeta) = 1$. As for $H = K^2$, where $K$ is the fundamental function of a Cartan space it is known that $\zeta$ is the unit normal vector field to $I_K$, we look for conditions on $H$ such that $\zeta$ to be the unit normal vector field for the indicatrix bundle of the Hamilton space $H^n = (M, H)$. For the geometry of the Cartan spaces we refer to the Ch. 6 in [7].

Let be

$$x^i = x^i(u^\alpha),\qquad p_i = p_i(u^\alpha),\ \alpha = 1, 2, \cdots, 2n-1 \tag{3.1}$$

a parametrization of the submanifold $I_H$. The local vector fields $\frac{\partial}{\partial u^\alpha}$ that form a basis of the tangent space to $I_H$ can be put in the form

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial p_i}{\partial u^\alpha} - N_{ij}(x, p)\frac{\partial x^j}{\partial u^\alpha}\right)\dot{\partial}^i . \tag{3.2}$$

If one derives the identity $H(x(u^\alpha),\ p(u^\alpha)) \equiv 1$ with respect to $u^\alpha$, one obtains

$$(\delta_i H)\frac{\partial x^i}{\partial u^\alpha} + \left(\dot{\partial}^i H\right)\left(\frac{\partial p_i}{\partial u^\alpha} - N_{ij}\frac{\partial x^j}{\partial u^\alpha}\right) \equiv 0. \tag{3.3}$$

On using (3.2) we see that $\zeta$ is normal to $I_H$ if and only if

$$G\left(\frac{\partial}{\partial u^\alpha}, \zeta\right) = \frac{1}{\varepsilon}\left(g^{ij}p_j\right)\left(\frac{\partial p_i}{\partial u^\alpha} - N_{ij}(x(u), p(u))\frac{\partial x^j}{\partial u^\alpha}\right) = 0 \tag{3.4}$$

for every $\alpha = 1, 2, \cdots, 2n-1$.

Comparing (3.3) with (3.4) it comes out that (3.4) holds if

$$\delta_i H = 0,\ \dot{\partial}^i H = f g^{ij} p_j,\ \text{for } f \text{ a smooth function on } T_0^* M. \tag{3.5}$$

The conditions (3.5) are quite complicated. We noticed them having in mind the case $H = K^2$, for $K$ the fundamental function of a Cartan space. In such a case, it is well known that $\delta_i K^2 = 0$ and from the equality $K^2 = g^{ij}(x,p)p_i p_j$ it follows that $\dot{\partial}^i K^2 = 2g^{ij}p_j$. The question is whether exist non-homogeneous Hamiltonians that satisfy (3.5).

We show now that the so-called $\varphi$-Hamiltonians introduced and studied by us in [4], fulfill the conditions (3.5).

Let $K^n = (M, K)$ be a Cartan space and $\varphi : \mathrm{R}_+ \to \mathrm{R}$ a function of class $C^\infty$. Assume that $\varphi$ has the properties:

$$\varphi'(t) \neq 0,\ \varphi'(t) + 2t\varphi''(t) \neq 0 \ \text{for}\ t \in \mathrm{Im}(K^2). \tag{3.6}$$

Then $H := \varphi(K^2)$ is a regular Hamiltonian on $T_0^* M$ called the $\varphi$-Hamiltonian associated to $K^n$.

As we have seen in [4] the Hamiltonians $H = \varphi(K^2)$ and $K^2$ define the same nonlinear connection. We have $\delta_i H = \varphi'(K^2)\,\delta_i K^2 = 0$. Hence the first condition (3.5) holds for any $\varphi$-Hamiltonian.

Let $\overset{\circ}{g}{}^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}$ be the metric tensor of $K^n$ and $g^{ij}(x,p)$ the metric tensor (2.1) of $H = \varphi(K^2)$. A direct calculation gives

$$g^{ij}(x,p) = \varphi'\left(\overset{\circ}{g}{}^{ij} + 2\frac{\varphi''}{\varphi'}\overset{\circ}{p}{}^i\,\overset{\circ}{p}{}^j\right) \tag{3.7}$$

where

$$\overset{\circ}{p}{}^i = \overset{\circ}{g}{}^{ij}(x,p)p_j = \frac{1}{2}\frac{\partial K^2}{\partial p_i}.$$

We have

$$g^{ij}p_j = \varphi'\left(1 + \frac{2\varphi''}{\varphi'}K^2\right)\overset{\circ}{p}{}^i = \frac{\varphi' + 2\varphi'' K^2}{2\varphi'}\frac{\partial H}{\partial p_i}$$

because of
$$\frac{\partial H}{\partial p_i} = \varphi'\frac{\partial K^2}{\partial p_i}.$$

Thus the second condition (3.5) holds with $f = \dfrac{2\varphi'}{\varphi' + 2\varphi'' K^2} \neq 0$.

Let us consider a Hamilton space $H^n = (M, H)$ such that $\zeta$ is the unit normal vector field of the indicatrix bundle $I_H$ defined by *H*.

We restrict to $I_H$ the elements of the triple $F_2$ and indicate this fact by a bar over those elements. We have

- $\overline{\xi} = \xi$ since $\xi$ is tangent to $I_H$,
- $\overline{\eta} = 0$ on $I_H$, since $\eta(X) = G(X, \zeta) = 0$ for any vector field $X$ tangent to $I_H$,
- $\overline{P}_2 = P - \omega \otimes \xi$ on $I_H$, because of $G(\overline{P}_2 X, \zeta) = G(PX, \zeta) = \eta(PX) = -\eta(X) = 0$ for any vector field $X$ tangent to $I_H$.

We have

**Theorem 3.1.** *The triple* $(\overline{P}_2, \overline{\xi}, \overline{\varpi})$ *defines a Riemannian almost paracontract structure on* $I_H$*, that is,*

*(i)* $\overline{\varpi}(\overline{\xi}) = 1,\ \overline{P}_2(\overline{\xi}) = 0,\ \overline{\varpi} \circ \overline{P}_2 = 0$

*(ii)* $\overline{P}_2^2 = I - \overline{\varpi} \otimes \overline{\xi}$ *on* $I_H$

*(iii)* $G(\overline{P}_2 X, \overline{P}_2 Y) = G(X, Y) - \overline{\varpi}(X)\overline{\varpi}(Y)$*, for any vector fields X, Y tangent to* $I_H$.

*Proof.* All the assertions follow from Theorems 2.2 and 2.3.

For $L = K^2$ we regain our results from [3]. Concluding, we have enlarged the set of Hamiltonians for which Theorem 3.1 holds good.

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Keywords and phrases: cotangent bundle, framed $f(3, -1)$ *-structures*, Hamilton spaces.