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On General Asymptotic Behaviour of Order Statistics with Random Index

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Abstract. This paper is devoted to the study of the general asymptotic behaviour of order statistics (extreme, intermediate and central terms) with random index. The case when the random index is assumed to be independent of the basic random variables (rv's) is studied as well as the case when the interrelation between the random sample size and the basic rv's is not restricted.

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1. Introduction

Let $X_1, X_2, \cdots, X_n$ be independent rv's each of which has the same continuous distribution function (df) $F(x)$. Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics of the rv's $X_1, X_2, \cdots, X_n$. Assume that $a_n > 0$ and b_n be suitable normalizing constants and ν_n be a positive integer valued rv. In many biological, agricultural and in some quality control problems it is almost impossible to have a fixed sample size because some observations always get lost for various reasons. Therefore, in applications the sample size n in $X_{r:n}, 1 \le r \le n$, itself is frequently a rv. Hence, much attention has been paid to weak convergence of $X_{r:\nu_n}$, $1 \le r \le \nu_n$, when properly normalized.

Here, the normalization is very important. Galambos [15,16] pointed out that if we allow normalization by $a_n > 0$ and b_n with the (same) random indices, then the normalizing constants may dominate both the conditions for convergence and the actual form of the limiting distribution. So, the only interesting weak convergence results are those when the normalizing constants are not random. The main aim of this paper is to investigate the weak convergence $\left(\xrightarrow[n]{w}\right)$ of the random sequence $\frac{X_{r_{\nu_n}:\nu_n} - b_n}{a_n}$, as $n \to \infty$, where $1 \le r_n < n$, and the df of the normalized ν_n weakly converges to a nondegenerate df. Many authors considered the preceding problem in the following cases:

(1) Extreme case $(r_n = r = \text{constant}$ or $r_n = n - r)$: See, for example [12], [14], [17], [18], [6], [4], [5], [9] and [11].

(2) Intermediate case $\left(\frac{r_n}{n} \underset{n}{\rightarrow} 0 \; \frac{r_n}{n} \underset{n}{\rightarrow} 1, \right.$ where $r_n \underset{n}{\rightarrow} \infty$ and $\left(\underset{n}{\rightarrow}\right)$ means convergence as $\left. n \rightarrow \infty\right)$: See, for example [21], [3], [7] and [8].

(3) Central case $\left(\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\rightarrow} 0,\ 0 < \lambda < 1\right)$: See, for example [22] and [19].

The paper is organized as follows: In Section 2, the above stated problem will be considered for the central case when the random index v_n is assumed to be independent of all basic rv's $X_1, X_2, \cdots, X_n$. Moreover, the study will be carried under each of the following exhaustive assumptions:

(i) $\sqrt{n}\,(\frac{r_n}{n} - \lambda) \underset{n}{\rightarrow} t,\ 0 \le \lambda \le 1, -\infty \le t \le \infty$.

(ii) $\sqrt{n}\,(\frac{r_n}{n} - \lambda)$ is bounded but does not tend to a limit, as $n \rightarrow \infty$.

In Section 3, by combining the results of Section 2 with the results concerning the above three cases, a general theorem for the limit df's of order statistics with random indices is established. Finally, Section 4 is devoted to the study the case when the interrelation between the sample size v_n and the basic rv's is not restricted.

2. Weak convergence of general central order statistics with random index

Throughout this paper the following abbreviations will be adopted $Y_{n:m} = \frac{X_{r_n:n} - b_m}{a_m}$, $n, m = 1, 2, \cdots$; $G_n(x) = P(X_{r_n:n} \le x)$; $G_n^*(x) = P(Y_{n:n} \le x)$, $\Psi_n^*(x) = P(Y_{v_n:n} \le x)$. Furthermore, let $\Phi(\cdot)$ stand for the standard normal df. When the rank sequence $\{r_n\}_n$ is assumed to satisfy the regular condition $\sqrt{n}\,(\frac{r_n}{n} - \lambda) \underset{n}{\rightarrow} 0$, Shokry [22], proved the following theorem.

Theorem 2.1. *Consider the following three conditions:*

(A) $$G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Phi\,(W_i^{(\beta)}(x)),\quad i \in \{1, 2, 3, 4\},$$

where $W_i^{(\beta)}(x)$ *has one and only one of the following possible limiting types:*

Type I : $W_1^{(\beta)}(x) = \begin{cases} -\infty, & x \le 0, \\ cx^{\beta}, & x > 0, c, \beta > 0, \end{cases}$

Type II : $W_2^{(\beta)}(x) = \begin{cases} -c|x|^{\beta}, & x \le 0 \\ \infty, & x > 0, c, \beta > 0, \end{cases}$

Type III : $W_3^{(\beta)}(x) = \begin{cases} -c_1|x|^{\beta}, & x \le 0, c_1 > 0, \\ c_2 x^{\beta}, & x > 0, c_2, \beta > 0, \end{cases}$

Type IV : $$W_4^{(\beta)}(x) = W_4(x) = \begin{cases} -\infty, & x \le -1, \\ 0, & -1 < x \le 1, \\ \infty, & x > 1. \end{cases} \qquad (2.1)$$

(B) $A_n(nx) = P\left(\frac{v_n}{n} \le x\right) \underset{n}{\overset{w}{\to}} A(x)$, *where* $A(x)$ *is a df such that* $A(+0) = 0$.

(C) $\Psi_n^*(x) \underset{n}{\overset{w}{\to}} \Psi(x) = \mathrm{E}\left(\Phi\left(\sqrt{Z}W_i^{(\beta)}(x)\right)\right)$, *where Z is a rv which is assumed to be distributed as* $A(x)$, *i.e.,* $\mathrm{E}\left(\Phi\left(\sqrt{Z}\, W_i^{(\beta)}(x)\right)\right) = \int_{-\infty}^{\infty} \Phi\left(\sqrt{Z}\, W_i^{(\beta)}(x)\right) dA(z)$.

The implications between the aforesaid conditions are

(1) $(A) \cap (B) \subset (C)$,
(2) $(A) \cap (C) \subset (B)$,
(3) $(B) \cap (C) \subset (A)$.

Remark 2.1. The implication (3) (in Theorem 2.1) is proved, by [22], only for the types $W_1^{(\beta)}$, $W_2^{(\beta)}$ and $W_3^{(\beta)}$.

We now consider the following three exhaustive cases:

Case 1: $\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\longrightarrow} t, \quad -\infty < t < \infty$.

Case 2: $\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\longrightarrow} \pm\infty$.

Case 3: $\sqrt{n}\left(\frac{r_n}{n} - \lambda\right)$ is bounded, but does not tend to a limit, as $n \to \infty$.

In the Case 1, the next theorem shows that Theorem 2.1 is still true (with only the obvious modifications).

Theorem 2.2. *Consider the following three conditions:*

(A) $G_n^*(x) \xrightarrow[n]{w} \Phi(W_i^{(\beta)}(x) + c_\lambda t)$, $i \in \{1,2,3,4\}$, *where* $\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\longrightarrow} t$,

$-\infty < t < \infty$, $0 < \lambda < 1$ *and* $c_\lambda = \dfrac{1}{\sqrt{\lambda(1-\lambda)}}$.

(B) $A_n(nx) \xrightarrow[n]{w} A(x)$, $A(x)$ *is a df such that* $A(+0) = 0$.

(C) $\Psi_n^*(x) \xrightarrow[n]{w} \Psi(x) = \mathrm{E}\left(\Phi\left(\sqrt{Z}\left(W_i^{(\beta)}(x) + tc_\lambda\right)\right)\right)$.

The implications between the conditions (A), (B) and (C) are

(1) $(A) \cap (B) \subset (C)$,

(2) $(A) \cap (C) \subset (B)$,

(3) $(B) \cap (C) \subset (A)$.

Remark 2.2. The implication (3) (in Theorem 2.1) will be proved only for the types $W_1^{(\beta)}$, $W_2^{(\beta)}$ and $W_3^{(\beta)}$.

We notice that the method of the proof of Theorem 2.2 is similar to the proof of Theorem 2.1 [22]. On the other hand, the proof of Theorem 2.1 is essentially based on a lemma which individually express a very interesting fact. This lemma states that the convergence in condition (A) (in Theorem 2.1) is uniform (although the limit df's may not be continuous). We will extend this lemma to cover the Case 1. Thus, in this case, Theorem 2.2 will be followed by the same method given in [22].

Lemma 2.1. *Assume that*

$$\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\longrightarrow} t, \quad -\infty < t < \infty,\ 0 < \lambda < 1. \tag{2.2}$$

Then,

$$G_n(x) = \Phi\left(\sqrt{n}\,\frac{F(x) - \frac{r_n}{n}}{\sqrt{\frac{r_n}{n}(1 - \frac{r_n}{n})}}\right) + R_n(x), \quad n = 1,2,3,\cdots,$$

where $R_n(x) \underset{n}{\longrightarrow} 0$, *uniformly with respect to* x.

Proof. It is well known that $G_n(x) = \frac{n!}{(r_n-1)!(n-r_n)!}\int_0^{F(x)} z^{r_n-1}(1-z)^{n-r_n}\,dz$. Hence, by putting $z = \frac{r_n-1}{n-1}(1+y\alpha_n)$, where

$$\alpha_n = \sqrt{\frac{n-r_n}{(r_n-1)(n-1)}} \underset{n}{\rightarrow} 0, \tag{2.3}$$

we obtain

$$G_n(x) = \frac{1}{\sqrt{2\pi}} \times \frac{\phi_{n-1}}{\phi_{r_n-1}\phi_{n-r_n}} \times \frac{n}{n-1} \int_{-\frac{1}{\alpha_n}}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} e^{-\frac{y^2}{2}+\theta_n}\,dy,$$

where $\theta_n = \frac{y^2}{2} + (r_n-1)\ln(1+y\alpha_n) + (n-r_n)\ln(1-\frac{r_n-1}{n-r_n}y\,\alpha_n)$ and $\phi_\alpha = \frac{a!}{a^a\sqrt{2\pi a}}$.
By virtue of (2.2) and (2.3), we get

$$\frac{1}{\alpha_n} - \sqrt{\frac{nr_n}{n-r_n}} \underset{n}{\rightarrow} 0 \tag{2.4}$$

and

$$\sqrt{\frac{(n-1)^3}{(n-r_n)(r_n-1)}} - \sqrt{\frac{n^3}{r_n(n-r_n)}} \underset{n}{\rightarrow} 0. \tag{2.5}$$

Now, consider the difference

$$\Delta_n = \frac{1}{\alpha_n}\left(\frac{n-1}{r_n-1}F(x)-1\right) - \sqrt{n}\left(\frac{F(x)-\frac{r_n}{n}}{\sqrt{\frac{r_n}{n}(n-r_n)}}\right)$$

$$= F(x)\left(\sqrt{\frac{(n-1)^3}{(n-r_n)(r_n-1)}} - \sqrt{\frac{n^3}{r_n(n-r_n)}}\right) - \left(\sqrt{\frac{(r_n-1)(n-1)}{n-r_n}} - \sqrt{\frac{nr_n}{n-r_n}}\right).$$

From (2.3), (2.4), (2.5) and the fact that $F(x) \leq 1$, we deduce that, $\Delta_n \underset{n}{\to} 0$, uniformly with respect to x. On the other hand, by simple calculations we can show that

$$\phi_a = 1 + \frac{23\overline{\theta}}{22(12a)} = 1 + O\left(\frac{1}{a}\right), \qquad \left|\overline{\theta}\right| < 1. \tag{2.6}$$

Using (2.6) and the fact that $\min(r_n, n - r_n) \underset{n}{\to} \infty$, we have

$$\frac{\phi_{n-1}}{\phi_{r_n-1}\,\phi_{n-r_n}} \underset{n}{\to} 1. \tag{2.7}$$

Now, we can write

$$\int_{-\frac{1}{\alpha_n}}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} e^{-\frac{y^2}{2}+\theta_n}\,dy = \sum_{i=1}^{4}(-1)^i I_n^{(i)}(x),$$

where

$$I_n^{(1)}(x) = I_n^{(1)} = \int_{-\infty}^{-\frac{1}{\alpha_n}} e^{\frac{-y^2}{2}}dy = \sqrt{2\pi}\,\Phi\left(-\frac{1}{\alpha_n}\right) \underset{n}{\to} 0, \tag{2.8}$$

$$I_n^{(2)}(x) = \int_{-\frac{1}{\alpha_n}}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} \left(e^{-\frac{y^2}{2}+\theta_n} - e^{-\frac{y^2}{2}}\right)dy,$$

$$I_n^{(3)}(x) = \int_{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)}^{\sqrt{n}\left((F(x)-\frac{r_n}{n})/\sqrt{\frac{r_n}{n(1-\frac{r_n}{n})}}\right)} e^{\frac{-y^2}{2}}\,dy \underset{n}{\to} 0 \tag{2.9}$$

and

$$I_n^{(4)}(x) = \sqrt{2\pi}\,\Phi\left(\sqrt{n}\left(\frac{F(x)-\frac{r_n}{n}}{\sqrt{\frac{r_n}{n}(1-\frac{r_n}{n})}}\right)\right).$$

Therefore, we get $G_n(x) = \frac{1}{\sqrt{2\pi}} I_n^{(4)}(x) + R_n(x)$, where

$$R_n(x) = \frac{1}{\sqrt{2\pi}} \times \frac{\phi_{n-1}}{\phi_{r_n-1}\,\phi_{n-r_n}} \times \frac{n}{n-1} \sum_{i=1}^{3} (-1)^i\, I_n^{(i)}(x)$$

$$+ \frac{1}{\sqrt{2\pi}} \times I_n^{(4)}(x) \times \left(\frac{n}{n-1} \times \frac{\phi_{n-1}}{\phi_{r_n-1}\,\phi_{n-r_n}} - 1 \right). \tag{2.10}$$

It remains to prove that $R_n(x) \underset{n}{\to} 0$, uniformly with respect to x. Using (2.7) and the fact that $\left| \frac{I_n^{(4)}(x)}{\sqrt{2\pi}} \right| \le 1$, the last term on the RHS of (2.10) tends uniformly to zero with respect to x, as $n \to \infty$. To prove $I_n^{(2)}(x) \underset{n}{\to} 0$, let us write $\phi(w_n) = (r_n - 1)\ln(1 + w_n) + (n - r_n)\ln(1 - \frac{r_n-1}{n-r_n} w_n)$, where $w_n = y\alpha_n$. Then, we get $\phi(w_n) - \frac{y^2}{2} + \theta_n$. By differentiating $\phi(w_n)$ with respect to w_n, we get $\phi'(w_n) = -\frac{(r_n-1)(n-1)}{n-r_n} \times \frac{w_n}{(1-w_n)(1-\frac{r_n-1}{n-r_n}w_n)}$, and $\phi''(w_n) = -\frac{(r_n-1)(n-1)}{n-r_n} \times \frac{1+\frac{r_n-1}{n-r_n}w_n^2}{(1+w_n)^2(1-\frac{r_n-1}{n-r_n}w_n)^2}$. When y changes from $-\frac{1}{\alpha_n}$ to $\frac{1}{\alpha_n}\frac{n-r_n}{r_n-1}$, the variable w_n changes from -1 to $\frac{n-r_n}{r_n-1}$, which implies $1 + w_n > 0$ and $1 - \frac{r_n-1}{n-r_n} w_n > 0$. Thus, $\phi'(w_n)$ and w_n have opposite signs. Moreover, $\phi''(w_n) < 0$. Therefore, for any $A > 0$, it is easy to show that

$$\int_{-\frac{1}{\alpha_n}}^{A} e^{\phi(w_n)}\,dy < \frac{1}{\alpha_n \phi'(-A\alpha_n)} \; e^{\phi(-A\alpha_n)} \tag{2.11}$$

and

$$\int_{A}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} e^{\phi(w_n)}dy < \frac{-1}{\alpha_n \phi'(A\alpha_n)} e^{\phi(A\alpha_n)}. \tag{2.12}$$

Using the Maclaurin's expansion for the function $\phi(w_n)$ we get

$$\phi(w_n) = \phi(0) + \frac{\phi'(0)}{1!} w_n + \frac{\phi''}{2!} w_n^2 \;(1 + o(1)), \quad (w_n \underset{n}{\to} 0),$$

where $|o(1)| < \frac{1}{2}$, when $|w_n| < \delta_1$. Therefore, if $|A\alpha_n| < \delta_1$, we get

$$\phi(A\alpha_n) < -\frac{(r_n-1)(n-1)}{n-r_n} \times \frac{\alpha_n^2 A^2}{4} = -\frac{A^2}{4}. \tag{2.13}$$

Moreover, $\phi'(w_n) = \phi''(0) + w_n(1+o(1))$, $w_n \underset{n}{\rightarrow} 0$. By choosing δ_2 such that $0 < \delta_2 \le \delta_1$, when $|w_n| < \delta_2$, we get $|\alpha_n \phi'(w_n)| > |w_n| \frac{|\phi''(0)|\alpha_n}{2}$. Therefore, if $|A\alpha_n| < \delta_2$, we get

$$|\alpha_n \phi'(A\alpha_n)| > |A\alpha_n| \frac{|\phi''(0)|\alpha_n}{2} = \frac{A\alpha_n|\phi''(0)|}{2} = \frac{A}{2}. \tag{2.14}$$

Now, by using (2.11), (2.12), (2.13) and (2.14) under the condition that $|A\alpha_n| < \delta_2$, we get

$$\int_{-\frac{1}{\alpha_n}}^{-A} e^{\phi(w_n)}\, dy + \int_{A}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} e^{\phi(w_n)} dy < \frac{4}{A} e^{-\frac{A^2}{4}}. \tag{2.15}$$

On the other hand, we can easily prove that

$$\int_{-\frac{1}{\alpha_n}}^{-A} e^{\frac{-y^2}{2}}\, dy + \int_{A}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} e^{-\frac{y^2}{2}} dy < \frac{4}{A} e^{-\frac{A^2}{2}}. \tag{2.16}$$

From (2.15) and (2.16), it follows that

$$\left| \int_{-\frac{1}{\alpha_n}}^{-A} \left(e^{\phi(w_n)} - e^{-\frac{y^2}{2}} \right) dy + \int_{A}^{\frac{1}{\alpha_n}(\frac{n-1}{r_n-1}F(x)-1)} \left(e^{\phi(w_n)} - e^{\frac{-y^2}{2}} \right) dy \right| < \frac{4}{A} e^{-\frac{A^2}{4}}. \tag{2.17}$$

For every $0 < \varepsilon < 1$, let $A = 4\sqrt{\ln \frac{4}{\varepsilon}}$. Then

$$\frac{4}{A} e^{-\frac{A^2}{4}} < \frac{\varepsilon}{4}. \tag{2.18}$$

We now show that for every $\varepsilon > 0$ there exists N such that

$$\int_{-A}^{A} e^{\frac{-y^2}{2}} \left| e^{\theta_n} - 1 \right| dy < \frac{\varepsilon}{5}, \text{ whenever } n > N. \tag{2.19}$$

For this purpose let us rewrite θ_n, in the following form

$$\theta_n = \frac{y^2}{2} + (r_n - 1)\ln(1 + y\alpha_n) + (n - r_n)\ln\left(1 - \frac{r_n - 1}{n - r_n} y\alpha_n\right)$$

$$= y^2 \frac{n - r_n}{n-1}\left[\frac{y\alpha_n}{3} - \frac{(y\alpha_n)^2}{4} + \frac{(y\alpha_n)^3}{5} + \cdots\right]$$

$$- y^2 \frac{r_n - 1}{n-1}\left[\frac{1}{3} y\alpha_n \left(\frac{r_n - 1}{n - r_n}\right) + \frac{1}{A}\alpha_n^2 y^2 \left(\frac{r_n - 1}{n - r_n}\right)^2 + \cdots\right].$$

Since $\alpha_n \underset{n}{\to} 0$ we can take N_1 large enough such that for $n > N_1$, we have $|\theta_n| < 1$ (note that $\theta_n \to 0$, as $\alpha_n \to 0$). Hence, it follows that

$$\int_{-A}^{A} e^{-\frac{y^2}{2}} \left| e^{\theta_n} - 1 \right| dy < 5 \max_{-A \le y \le A} |\theta_n| < 5 \frac{\varepsilon}{25} = \frac{\varepsilon}{5}.$$

Therefore, in view of (2.17), (2.18) and (2.19), for every $\varepsilon > 0$, there exists N_2 such that

$$\left| I_n^{(2)} \right| < \varepsilon \quad \text{whenever} \quad n > N_2. \tag{2.20}$$

Finally, from (2.8), (2.9), (2.20) and (2.10), we get $\lim_{n\to\infty} R_n(x) = 0$. This completes the proof of the lemma.

We discuss the Cases 2 and 3 throughout an example, which shows that either $\Psi(x)$ does not exist or it is degenerate.

Example 2.1. Under the conditions of the Case 2, Wu [23] has shown that the class of all possible limit nondegenerate df's of the term $Y_{n:n}$ contains only the types $\Phi(V_i^{(\beta)}(x))$, $i = 1, 2, 3$, where

Type I: $V_1(x) = V_1^{(\beta)}(x) = x, \ \forall \ x;$

Types II: $V_2^{(\beta)}(x) = \begin{cases} -\beta \ln|x|, & x \le 0, \\ \infty, & x > 0; \end{cases}$

Types III: $V_3^{(\beta)}(x) = \begin{cases} -\infty, & x \le 0, \\ \beta \ln x, & x > 0; \end{cases}$

Where β is some positive constant. Moreover, $G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Phi(V_i^{(\beta)}(x))$, $i \in \{1,2,3\}$, if and only if $U_{n,n}(x) = \sqrt{n}(F(a_n x + b_n) - \frac{r_n}{n})c_\lambda \rightarrow V_i^{(\beta)}(x)$, $c_\lambda = \frac{1}{\sqrt{\lambda(1-\lambda)}}$, where $a_n > 0$ and b_n are suitable normalizing constants. Assume now that the positive integer rv ν_n is independent of all basic rv's $X_1, X_2, \cdots, X_n$. Furthermore, let $A_n(nx) \underset{n}{\overset{w}{\rightarrow}} A(x)$, $x \geq 0$ and $G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Phi(V_i^{(\beta)}(x))$, $i \in \{1,2,3\}$. Then by using the total probability rule, we get

$$\Psi_n^*(x) \sim \int_0^\infty \Phi(U_{m,n}(x))\, dA_n(m),$$

where $U_{m,n}(x) = \sqrt{m}\,(F(a_n x + b_n - \frac{r_m}{m})c_\lambda$. However, we can write $U_{m,n}(x) = \sqrt{\frac{m}{n}}\left(U_{n,n}(x) + \sqrt{n}\, c_\lambda (\frac{r_n}{n} - \frac{r_m}{m})\right)$. Thus, if we put $m = nz$, we get $U_{n,n}(x) \underset{n}{\rightarrow} V_i^{(\beta)}(x)$, $i \in \{1,2,3\}$ and $\sqrt{\frac{m}{n}} \underset{n}{\rightarrow} \sqrt{z}$. On the other hand, in view of Corollary 2 of [23], there exists a subsequence $\{n_s\}_s$ of natural numbers, such that $\frac{n_{s+1}}{n_s} \underset{n}{\rightarrow} L > 1$. Moreover, the sequence $\left\{\sqrt{n_s}\,(\frac{r_{n_s}}{n_s} - r_{n_{s+1}})\right\}$ possesses a positive limit point, which has infinite value. Therefore for all $z > 1$, we get

$$\lim_{n\to\infty} \sup \sqrt{n}\left(\frac{r_n}{n} - \frac{r_{nz}}{nz}\right) \geq \lim_{n_s\to\infty} \sup \sqrt{n_s}\left(\frac{r_{n_s}}{n_s} - \frac{r_{n_{s+1}}}{n_{s+1}}\right) = \infty,$$

where (and in the sequel), if ℓ is not integer, r_ℓ, is understood as $r_{[\ell]}$ and as usual $[\ell]$ denotes the integer part of ℓ. The above relation implies

$$\lim_{n\to\infty} \sup \sqrt{n}\left(\frac{r_n}{n} - \frac{r_{nz}}{nz}\right) = \infty, \text{ for all } z > 1, \tag{2.21}$$

which in turn shows that $\Psi(x)$, under our assumptions, either does not exist or is degenerate. Clearly, this example is still true, when $\sqrt{n}\,(\frac{r_n}{n} - \lambda) \underset{n}{\rightarrow} -\infty$ and also in the Case 3.

3. General asymptotic behaviour of order statistics with random index which is independent of the basic rv's

Let $a_n > 0$ and b_n be suitable normalizing constants and v_n be a positive integer valued rv such that

$$\Psi_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Psi(x), \tag{3.1}$$

where $\Psi(x)$ is a nondegenerate df. Let K_{v_n} be the class of all nondegenerate limits df's $\Psi(x)$ in (3.1). The following theorem is obtained by unifying the results of Section 2 and the known results given in the three cases in Section 1. This theorem characterizes the class K_{v_n} for any order statistics with a random index, which is assumed to be independent of the basic rv's.

Theorem 3.1. *For any nondegenerate df* $\Psi(x)$, $\Psi(x) \in K_{v_n}$ *if and only if one of the conditions (I), (II) and (III) holds*

$$(I)\begin{cases} (i) & r_n = r = \textit{constant (extreme case)}. \\ (ii) & G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Gamma_r(U_i^{(\beta)}(x)), \\ & \textit{where } \Gamma_r(\cdot) \textit{ is the incomplete Gamma function.} \\ (iii) & A_n(nx) \underset{n}{\overset{w}{\rightarrow}} A(x),\ A(x) \textit{ is a df such that } A(+0) = 0. \end{cases}$$

The limit df $\Psi(x)$, *has the form* $\Psi(x) = \mathrm{E}(\Gamma_r(ZU_i^{(\beta)}))$, $i \in \{1, 2, 3\}$, *where Z is a rv, which is distributed as* $A(x)$, $U_1^{(\beta)}(x) = x,\ x > 0$; $U_2^{(\beta)}(x) = (-x)^{-\beta},\ x < 0$; *and* $U_3^{(\beta)}(x) = e^x$, $-\infty < x < \infty$, $\beta > 0$.

$$(II)\begin{cases} (i) & \sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \underset{n}{\rightarrow} t,\ -\infty < t < \infty,\ 0 < \lambda < 1 \textit{ (central case)}. \\ (ii) & G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Phi(W_i^{(\beta)}(x) + tc_\lambda), \\ (iii) & A_n(nx) \underset{n}{\overset{w}{\rightarrow}} A(x),\ A(x) \textit{ is a df such that } A(+0) = 0. \end{cases}$$

The limit df $\Psi(x)$ *has the form* $\Psi(x) = \mathrm{E}(\Phi(\sqrt{Z}(W_i^{(\beta)}(x) + tc_\lambda)))$, *where* $W_i^{(\beta)}(x)$, $i = 1, 2, 3, 4$ *are defined in* (2.1).

$$(III)\begin{cases} (i) & \sqrt{r_{n+z_n}}-\sqrt{r_n} \ \underset{n}{\overset{w}{\rightarrow}} \ \frac{\alpha\ell\delta}{2}, \\ & \textit{for any sequence of integer values } \{z_n\} \textit{ for which } \frac{z_n}{n^{1-\frac{\alpha}{2}}} \underset{n}{\rightarrow} \delta, \\ & 0<\alpha<1,\ \ell>0 \textit{ and } \delta \textit{ is any real number (intermediate case)}. \\ (ii) & G_n^*(x) \underset{n}{\overset{w}{\rightarrow}} \Phi(V_i^{(\beta)}(x)). \\ (iii) & A_n(n^{1-\frac{\alpha}{2}}x+n) \underset{n}{\overset{w}{\rightarrow}} A(x) \textit{ is a df}. \end{cases}$$

The limit df $\Psi(x)$ *has the form*

$$\Psi(x)=\mathrm{E}(\Phi(V_i^{(\beta)}(x)+Z\ell(1-\alpha))). \tag{3.2}$$

If r_n *does not satisfy any of the conditions (i-I), (i-II) and (i-III), then* $\psi(x)$ *can only have a degenerate type or does not exist.*

In most of the sampling techniques, when we have to consider the sample size as a rv, this rv will have the same df for all order statistics under consideration (extreme, intermediate and central terms). Therefore, we can formulate the following result.

Theorem 3.2. *If we have a sample of random size and intermediate term with rank sequence satisfying the condition (i-III) (Chibisov's condition, see* [13]*) such that the relations (ii-III) and (iii-III) of Theorem* 3.1 *hold with nondegenerate df* $\Phi(V_i^{(\beta)}(x))$, $i\in\{1,2,3\}$ *and* $A(x)$, *then* (3.2) *will be satisfied with nondegenerate df* $\Psi(x)$. *In the same time, the df's of all extreme and central terms can only weakly converge to the df's* $\Gamma_r(U_i^{(\beta)}(x))$, $i\in\{1,2,3\}$ *and* $\Phi(W_i^{(\beta)}(x)+c_\lambda t)$, $i\in\{1,2,3,4\}$, *respectively. On the other hand, the convergence of the df's for extreme or central terms to nondegenerate df's (such that the relations (ii-I) and (iii-I) or the relations (ii-II) and (iii-II) are satisfied with nondegenerate df's) implies the nonconvergence of all intermediate terms.*

Proof. Assume that, we have a random index v_n for which $A_n(n^{1-\frac{\alpha}{2}}x+n)\underset{n}{\overset{w}{\rightarrow}}A(x)$, where $A(x)$ is a nondegenerate df. Let r_n be a rank sequence satisfying Chibisov's condition, i.e., $r_n\sim\ell^2n^{\frac{\alpha}{2}}$, $0<\alpha<1$. Since $\frac{n}{n^{1-\frac{\alpha}{2}}}\underset{n}{\rightarrow}\infty$, then by using Lemma 4.1.1 in [15] we get

$$A(nx)\underset{n}{\overset{w}{\rightarrow}}\begin{cases}0, & x\le 1,\\ 1, & x>1.\end{cases} \tag{3.3}$$

It is easy to show that (3.3) is the necessary and sufficient condition to get $\Psi(x) = \Gamma_r(U_i^{(\beta)}(x))$, in (I) and $\Psi(x) = \Phi(W_i^{(\beta)}(x) + c_\lambda t)$, in Theorem 2.1. The second part of the theorem follows from (I), Theorem 2.1 and [3] (see the example). This complete the proof of the theorem.

Example 3.1. Consider the rth order statistic from a sample of size n obtained from the uniform distribution on the interval $(0,1)$ with $1 < r_n < n$, and $\min(r_n, n - r_n) \underset{n}{\to} \infty$. It is well known that (see [2])

$$P(\alpha_n X_{r_n:n} + \beta_n \le x) \overset{w}{\underset{n}{\to}} \Phi(x),$$

where $\alpha_n = (\frac{r_n(n-r_n)}{n^3})^{-\frac{1}{2}}$ and $\beta_n = -\frac{r_n}{n} \Big/ \frac{r_n(n-r_n)^{\frac{1}{2}}}{n^3}$. If we have a positive integer valued rv v_n such that $A_n(nx) \overset{w}{\underset{n}{\to}} A(x)$, then $P(\alpha_n X_{r_{v_n}:v_n} + \beta_n \le x) \overset{w}{\underset{n}{\to}} \mathrm{E}(\Phi(\sqrt{Z}x))$, holds only for the order statistics $X_{r_n:n}$, for which there exist λ and t such that $-\infty < t < \infty$, $0 < \lambda < 1$ and $\sqrt{n}\,(\frac{r_n}{n} - \lambda) \underset{n}{\to} t$. If we have a rv v_n such that, $A(\frac{r_n}{\sqrt{n}} x + r_n) \overset{w}{\underset{n}{\to}} A(x)$, then

$$P(\alpha_n X_{r_{v_n}:v_n} + \beta_n \le x) \overset{w}{\underset{n}{\to}} \begin{cases} \mathrm{E}(\Phi(Z(1-\alpha)\ell + x)), & r \sim \ell^2 n^{\frac{\alpha}{2}}, \\ \Phi(x), & \sqrt{n}\,(\frac{r_n}{n} - \lambda) \underset{n}{\to} t, \ -\infty < t < \infty. \end{cases}$$

4. General asymptotic behaviour of order statistics with general random index

When the interrelation between the random index and the basic rv's is not restricted, parallel theorems of Theorems 3.1, 3.2 may be proved by replacing the conditions (*iii*)–(I), (II), (III) by stronger conditions. Namely, the weak convergence of the df's $A_n(nx)$ and $A_n(n^{1-\frac{\alpha}{2}}x + n)$ must be replaced respectively by the convergence in probability of the rv's $\frac{v_n}{n}$ and $\frac{v_n - n}{n^{1-\frac{\alpha}{2}}}$ to a positive rv Z. For maximum order statistics (part (I)). The parallel theorem of Theorem 3.1 is proved in [15] (Theorem 6.2.1). This result is extended to the extreme order statistics by [6]. For the intermediate case, under Chibisov condition, this parallel result is obtained by [3] and [7], [8]. Finally, [1] studied the central case with regular rank and general random index. However, the key ingredient of the proof of this parallel result (for the extreme, the intermediate and the central cases) is to prove the mixing property, due to Rényi (see, [10]) of the sequence of order statistics under consideration. In the sense of Rényi a sequence $\{X_n\}$ of rv's is called mixing if for any event ☜ of positive probability, the conditional distribution function of $\{X_n\}$, under the condition ε, converges weakly to a nondegenerate df, which does not depend on ☜, as $n \to \infty$. In Theorem 3.1 of [10], any sequence of order statistics with general variable rank ($\min(r_n, n - r_n) \underset{n}{\to} \infty$) was proved to be mixing. Therefore, when the interrelation between the random index and the basic rv's is not restricted, Theorem 3.1 can be extended to the Case 1. However, we can combine the two parallel theorems to obtain a general result, which may be useful in practical purpose.

Theorem 4.1. *Let* $A_n(x)$ *be any general sequence of df's (the interrelation between these df's and* $G_n^*(x)$ *is not restricted). Moreover, assume that one of the conditions (iii)–(I), (II) and (III). Then Theorem* 3.1 *will be satisfied with the class* $K_{v_n^*}$, *where* $v_n^* = v_n^*(\omega) = \inf\{x : \omega \leq A_n(nx)\}$, $\forall \omega \in (0,1)$, *in the extreme and the central cases and* $v_n^* = v_n^*(\omega) = \inf\{x : \omega \leq A_n(n^{1-\frac{\alpha}{2}}x + n)\}$, $\forall \omega \in (0,1)$, *in the intermediate case.*

Proof. The proof is immediately followed by Theorem 3.1 and Skorohod's representation theorem (see [20]).

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