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α-I-Preirresolute Functions and β-I-Preirresolute Functions

[1]A. AÇIKGÖZ, [2]Ş. YÜKSEL AND [3]T. NOIRI
[1, 2]Selcuk University, Department of Mathematics
42031 Campüs-Konya, Turkey
[3]Department of Mathematics, Yatsushiro College of Techology
Yatsushiro, Kumamoto, 866-8501, Japan
[2]syuksel@selcuk.edu.tr, [3]noiri@as.yatsushiro-nct.ac.jp

Abstract. The purpose of this paper is to introduce two new classes of functions called α-I-preirresolute functions and β-I-preirresolute functions in ideal topological spaces. Some properties and several characterisations of these types of functions are obtained. Also, we investigate the relationships between these classes of functions and other classes of non-continuous functions.

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1. Introduction

Yüksel *et al.* [17] introduced the notions of α-I-irresolute, α-pre-I-continuous and almost α-I-irresolute functions in ideal topological spaces. The purpose of the present paper is to introduce and investigate the notions of new classes of functions, namely α-I-preirresolute functions and β-I-preirresolute functions, and to give several characterizations and their properties. Relations between these types of functions and other classes of functions are obtained. The new class of α-I-preirresolute functions is stronger than pre-I-irresolute functions. The new class of β-I-preirresolute functions, which is stronger than almost α-I-irresolute functions [17], is a generalization of pre-I-irresolute functions.

2. Preliminaries

Throughout this paper $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A, respectively. Let (X,τ) be a topological space and let I an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions : (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A \subset X$, $A^*(I) = \{x \in X | U \cap A \notin I$ for

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each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ [12]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. X^* is often a proper subset of X. The hypothesis $X = X^*$ [10] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [16]. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I, \tau) = \{U \backslash I : U \in \tau \text{ and } I \in I\}$, but in general $\beta(I, \tau)$ is not always a topology [11]. Additionally, $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

We recall some known definitions.

Definition 2.1. *A subset S of a topological space (X, τ) is said to be α-open* [15] *(resp. pre-open* [13]*, β-open* [1]*) if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(S)), S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$).*

Definition 2.2. *A subset A of an ideal topological space (X, τ, I) is said to be α-I-open* [8] *(resp. pre-I -open* [4]*, β-I-open* [8]*) if $A \subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}^*(A)), S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(S)))$). The family of all α-I-open (resp. pre-I-open, β-I -open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha IO(X)$ (resp. $PIO(X)$, $\beta IO(X)$). The intersection of all preclosed sets containing a subset S is called the preclosure* [7] *of S and is denoted by $\mathrm{pCl}(S)$; the union of all preopen sets contained in S is called the preinterior* [14] *of S and is denoted by $\mathrm{pInt}(S)$.*

Definition 2.3. [17] *A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be α-I-irresolute (resp. almost α-I-irresolute) if $f^{-1}(V)$ is α-I-open (resp. β-I-open) in X for every α-open set V of Y.*

Definition 2.4. [17] *A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be α-pre-I-continuous if $f^{-1}(V)$ is pre-I-open in X for every α-open set V of Y.*

Definition 2.5. *A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be pre-I-irresolute (resp. α-I-preirresolute, β-I-preirresolute) if $f^{-1}(V)$ is pre-I-open (resp. α-I-open, β-I-open) in X for every preopen set V of Y.*

From the definitions stated above, we obtain the following diagram:

$$\begin{array}{ccccc}
\alpha\text{-I-preirresolute} & \longrightarrow & \text{pre-I-irresoluteness} & \longrightarrow & \beta\text{-I-preirresoluteness} \\
\downarrow & & \downarrow & & \downarrow \\
\alpha\text{-I-irresoluteness} & \longrightarrow & \alpha\text{-pre-I-continuity} & \longrightarrow & \text{almost } \alpha\text{-I-irresoluteness}
\end{array}$$

Remark 2.1. However, converses of the above implications are not true, in general, by [17, Examples 1.1, 1.2 and 1.3].

3. α-I-preirresolute functions

Theorem 3.1. *For a function $f : (X, \tau, I) \to (Y, \nu)$, the following are equivalent:*

(a) *f is α-I-preirresolute;*

(b) *For each $x \in X$ and each preopen set V of Y containing $f(x)$, there exists an α-I-open set U of X containing x such that $f(U) \subset V$;*

(c) $f^{-1}(V) \subset \text{Int}(CI^*(\text{Int}(f^{-1}(V))))$ *for every preopen set* V *of* Y*;*
(d) $f^{-1}(F)$ *is* α*-I-closed in* X *for every preclosed set* F *of* Y*;*
(e) $\text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(B)))) \subset f^{-1}(\text{pCl}(B))$ *for every subset* B *of* Y*;*
(f) $f(\text{Cl}(\text{Int}^*(\text{Cl}(A)))) \subset \text{pCl}(f(A))$ *for every subset* A *of* X*.*

Proof.
(a) $\Rightarrow$ (b): Let $x \in X$ and V be any preopen set of Y containing $f(x)$. By Definition 2.5 $f^{-1}(V)$ is α-I-open in X and contains x. Set $U = f^{-1}(V)$, then U is an α-I-open subset of X containing x and $f(U) \subset V$.

(b)$\Rightarrow$(c): Let V be any preopen set of Y and $x \in f^{-1}(V)$. By (b), there exists an α-I-open set U of X containing x such that $f(U) \subset V$. Thus, we have

$$x \in U \subset \text{Int}(\text{Cl}^*(\text{Int}(U))) \subset \text{Int}(\text{Cl}^*(\text{Int}(f^{-1}(V))))$$

and hence

$$f^{-1}(V) \subset \text{Int}(\text{Cl}^*(\text{Int}(f^{-1}(V)))).$$

(c)$\Rightarrow$(d): Let F be any preclosed subset of Y. Set $V = Y - F$, then V is preopen in Y. By (c), we obtain $f^{-1}(V) \subset \text{Int}(\text{Cl}^*(\text{Int}(f^{-1}(V))))$ and hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is α-I-closed in X.

(d)$\Rightarrow$(e): Let B be any subset of Y. Since $\text{pCl}(B)$ is a preclosed subset of Y, then $f^{-1}(\text{pCl}(B))$ is α-I-closed in X and hence

$$\text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(\text{pCl}(B))))) \subset f^{-1}(\text{pCl}(B)).$$

Therefore, we obtain $\text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(B)))) \subset f^{-1}(\text{pCl}(B))$.

(e)$\Rightarrow$(f): Let A be any subset of X. By (e), we have

$$\text{Cl}(\text{Int}^*(\text{Cl}(A))) \subset \text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(f(A))))) \subset f^{-1}(\text{pCl}(f(A)))$$

and hence $f(\text{Cl}(\text{Int}^*(\text{Cl}(A)))) \subset \text{pCl}(f(A))$.

(f)$\Rightarrow$(a): Let V be any preopen subset of Y. Since $f^{-1}(Y - V) = X - f^{-1}(V)$ is a subset of X and by (f), we obtain

$$\begin{aligned} f(\text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(Y - V))))) &\subset \text{pCl}(f(f^{-1}(Y - V))) \\ &\subset \text{pCl}(Y - V) \\ &= Y - \text{pInt}(V) = Y - V \end{aligned}$$

and hence

$$\begin{aligned} X - \text{Int}(\text{Cl}^*(\text{Int}(f^{-1}(V)))) &= \text{Cl}(\text{Int}^*(\text{Cl}(X - f^{-1}(V)))) \\ &= \text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(Y - V)))) \\ &\subset f^{-1}(f(\text{Cl}(\text{Int}^*(\text{Cl}(f^{-1}(Y - V)))))) \\ &\subset f^{-1}(Y - V) \\ &= X - f^{-1}(V). \end{aligned}$$

Therefore, we have $f^{-1}(V) \subset \text{Int}(\text{Cl}^*(\text{Int}(f^{-1}(V))))$ and hence $f^{-1}(V)$ is α-I-open in X. Thus, f is α-I-preirresolute. □

Lemma 3.1 (Chae et al. [3], El-Deeb et al. [7] and Abd El-Monsef et al. [1]). *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces and U_{λ_i} be a nonempty subset of X_{λ_i} for each $i = 1, 2, \ldots, n$. Then $U = \prod_{\lambda \neq \lambda i} X_\lambda \times \prod_{i=1}^{n} U_{\lambda_i}$ is a nonempty α-open [3] (resp. preopen [7], β-open [1]) subset of $\prod X_\lambda$ if and only if U_{λ_i} is α-open (resp. preopen, β-open) in X_{λ_i} for each $i = 1, 2, \ldots, n$.*

Theorem 3.2. *A function $f : (X, \tau, I) \to Y$ is α-I-preirresolute if the graph function $g : (X, \tau, I) \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is α-I-preirresolute.*

Proof. Let $x \in X$ and V be any preopen set of Y containing $f(x)$. Then $X \times V$ is a preopen set of $X \times Y$ by Lemma 3.1 and contains $g(x)$. Since g is α-I-preirresolute, there exists an α-I-open set U of X containing x such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus f is α-I-preirresolute. □

Theorem 3.3. *If a function $f : (X, \tau, I) \to \prod Y_\lambda$ is α-I-preirresolute, then $p_\lambda \circ f : (X, \tau, I) \to Y_\lambda$ is α-I-preirresolute for each $\lambda \in \Lambda$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.*

Proof. Let V_λ be any preopen set of Y_λ. Since P_λ is continuous and open, it is preirresolute [13, Theorem 3.4]. Therefore, $P_\lambda^{-1}(V_\lambda)$ is preopen in $\prod Y_\lambda$. Since f is α-I-preirresolute, then $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda \circ f)^{-1}(V_\lambda)$ is α-I-open in X. Hence $P_\lambda \circ f$ is α-I-preirresolute for each $\lambda \in \Lambda$. □

Theorem 3.4. *If $f : (X, \tau, I) \to (Y, \nu)$ is α-I-preirresolute and A is an α-I-open subset of X, then the restriction $f/A : A \to Y$ is α-I-preirresolute.*

Proof. Let V be any preopen set of Y. Since f is α-I-preirresolute, then $f^{-1}(V)$ is α-I-open in X. Since A is α-I-open in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is α-I-open in A [2, Theorem 3.1]. Hence f/A is α-I-preirresolute. □

Theorem 3.5. *Let $f : (X, \tau, I) \to (Y, \nu)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by α-I-open sets of (X, τ, I). Then f is α-I-preirresolute if and only if $f/A_\lambda : A_\lambda \to Y$ is α-I-preirresolute for each $\lambda \in \Lambda$.*

Proof. Necessity. This follows from Theorem 3.4.

Sufficiency. Let V be any preopen set of Y. Since f/A_λ is α-I-preirresolute, $(f/A_\lambda)^{-1}(V)$ is α-I-open in A_λ. Since A_λ is α-I-open in X, $(f/A_\lambda)^{-1}(V)$ is α-I-open in X for each $\lambda \in \Lambda$ [2, Theorem 3.2]. Therefore,

$$f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}$$

is α-I-open in X because the union of α-I-open sets is an α-I-open set [2, Proposition 3.2(2)]. Hence f is α-I-preirresolute. □

Theorem 3.6. *Let $f : (X, \tau, I) \to (Y, \nu)$ and $g : (Y, \nu) \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is α-I-preirresolute if f is α-I-preirresolute and g is preirresolute.*

Proof. Let W be any preopen subset of Z. Since g is preirresolute, $g^{-1}(W)$ is preopen in Y. Since f is α-I-preirresolute, then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is α-I-open in X and hence $g \circ f$ is α-I-preirresolute. □

4. β-I-preirresolute functions

Lemma 4.1. *Let (X, τ, I) be an ideal topological space.*

(1) *If $A \in \tau$ and $B \in \beta IO(X)$, then $A \cap B \in \beta IO(A)$.*
(2) *If $A \in \alpha IO(X)$ and $B \in \beta IO(X)$, then $A \cap B \in \beta IO(X)$.*

Proof.

(1): This property is shown in [9, Theorem 4.3].

(2): We have

$$\begin{aligned} A \cap B &\subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(B))) \\ &\subset \mathrm{Cl}[\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \cap \mathrm{Int}(\mathrm{Cl}^*(B))] \\ &= \mathrm{Cl}(\mathrm{Int}[\mathrm{Cl}^*(\mathrm{Int}(A)) \cap \mathrm{Int}(\mathrm{Cl}^*(B))]) \\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*[\mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}^*(B))])) \\ &= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}[\mathrm{Int}(A) \cap \mathrm{Cl}^*(B)]))) \\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(A \cap B))))) \\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A \cap B))). \end{aligned}$$

□

Lemma 4.2. *If $A \subset X_o \subset X$, $X_o \in \tau$ and $A \in \beta IO(X_o)$, then $A \in \beta IO(X)$.*

Proof.

$$\begin{aligned} A \subset \mathrm{Cl}_{X_o}(\mathrm{Int}_{X_o}(\mathrm{Cl}^*_{X_o}(A))) &= \mathrm{Cl}(\mathrm{Int}_{X_o}(\mathrm{Cl}^*_{X_o}(A))) \cap X_o \\ &\subset \mathrm{Cl}(\mathrm{Int}_{X_o}(\mathrm{Cl}^*_{X_o}(A))) \\ &= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*_{X_o}(A))) \\ &= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A) \cap X_o)) \\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A))). \end{aligned}$$

□

Theorem 4.1. *For a function $f : (X, \tau, I) \to (Y, \nu)$, the following are equivalent:*

(a) *f is β-I-preirresolute;*
(b) *For each $x \in X$ and each preopen set V of Y containing $f(x)$, there exists a β-I-open set U of X containing x such that $f(U) \subset V$;*
(c) *$f^{-1}(V) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(f^{-1}(V))))$ for every preopen set V of Y;*
(d) *$f^{-1}(F)$ is β-I-closed in X for every preclosed of F of Y;*
(e) *$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(f^{-1}(B)))) \subset f^{-1}(\mathrm{pCl}(B))$ for every subset B of Y;*
(f) *$f(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A)))) \subset \mathrm{pCl}(f(A))$ for every subset A of X.*

Theorem 4.2. *A function $f : (X, \tau, I) \to Y$ is β-I-preirresolute if the graph function $g : (X, \tau, I) \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is β-I-preirresolute.*

Theorem 4.3. *If a function $f : (X, \tau, I) \to \prod Y_\lambda$ is β-I-preirresolute, then $P_\lambda \circ f : (X, \tau, I) \to Y_\lambda$ is β-I-preirresolute for each $\lambda \in \Lambda$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.*

Theorem 4.4. *If $f : (X, \tau, I) \to (Y, \nu)$ is β-I-preirresolute and A is an open subset of X, then restriction $f/A : A \to Y$ is β-I-preirresolute.*

Proof. Let V be any preopen set of Y. Since f is β-I-preirresolute, then $f^{-1}(V)$ is β-I-open in X. Since A is open in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is β-I-open in A by Lemma 4.1(1). Hence f/A is β-I-preirresolute. □

Theorem 4.5. *Let $f : (X, \tau, I) \to (Y, \nu)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by open sets of (X, τ, I). Then f is β-I-preirresolute if and only if $f/A_\lambda : A_\lambda \to Y$ is β-I-preirresolute for each $\lambda \in \Lambda$.*

Proof. Let V be any preopen set of Y. Since f/A_λ is β-I-preirresolute, $(f/A_\lambda)^{-1}(V)$ is β-I-open in A_λ. Since A_λ is open in X, then $(f/A_\lambda)^{-1}(V)$ is β-I-open in X for each $\lambda \in \Lambda$ by Lemma 4.2. Therefore,

$$f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}$$

is β-I-open in X because the union of β-I-open sets is a β-I-open set [9]. □

Theorem 4.6. *Let $f : (X, \tau, I) \to (Y, \nu)$ and $g : (Y, \nu) \to Z$ be functions. Then the composition $gof : X \to Z$ is β-I-preirresolute if f is β-I-preirresolute and g is preirresolute.*

Proof. The proof is similar to that of Theorem 3.6 and is thus omitted. □

We recall that a subset A of X is said to be *τ^*-dense* [5] (resp. **-dense-in-itself* [10], **-perfect* [10]) if $\mathrm{Cl}^*(A) = X$ (resp. $A \subset A^*$, $A = A^*$). A subset of X is said to be *I-locally closed* if it is the intersection of an open subset and a *-perfect subset of X [6].

We obtain the following theorem from the above definitions.

Theorem 4.7. *For a space (X, τ, I), the following are equivalent:*

(a) *Every *-dense-in-itself subset is pre-I-open.*
(b) *Every *-perfect subset is open.*

Proof.
(a)⇒(b): Let $A \subset X$ be *-perfect. By hypothesis, A is pre-I-open and hence $A \subset \mathrm{Int}(\mathrm{Cl}^*(A)) = \mathrm{Int}(A)$. Thus A is open.

(b)⇒(a): Let $A \subset X$ be *-dense-in-itself. Then $A \subset A^*$ and $A^* = \mathrm{Cl}^*(A)$. On the other hand, $A^* \subset (A^*)^* \subset A^*$ and hence $A^* = (A^*)^*$. Consequently, we have $(\mathrm{Cl}^*(A))^* = \mathrm{Cl}^*(A)$. Then $\mathrm{Cl}^*(A)$ is *-perfect. By hypothesis, $\mathrm{Cl}^*(A)$ is open, hence $A \subset \mathrm{Cl}^*(A) = \mathrm{Int}(\mathrm{Cl}^*(A))$. Thus A is pre-I-open. □

Now, we define the following.

Definition 4.1. *A subset of X is said to be co*-locally closed if it is the union of an open subset and a *-perfect subset of X.*

Definition 4.2. *A space (X, τ, I) is I-submaximal if every subset of X is I-locally closed.*

Theorem 4.8. *For a space (X, τ, I), the following statements are equivalent:*

(a) *X is an I-submaximal space,*
(b) *every subset of X is co*-locally closed,*
(c) *every subset A of X, for which A^* is empty, is open,*
(d) $\mathrm{Cl}^*(A)\backslash A$ *is closed for every subset A of X,*
(e) *every τ^*-dense subset of X is open.*

Proof. (e)$\Leftrightarrow$ (d)$\Rightarrow$(a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) are immediate. □

Proposition 4.1. *Every submaximal space is I-submaximal space.*

Proof. Let $A \subset X$ be τ^*-dense. Then $X = \mathrm{Cl}^*(A)$. Since $\tau \subset \tau^*$ and X is submaximal, then A is open in X. By Theorem 4.8 (e), X is I-submaximal space. □

The converse in the proposition above is not necessarily true as shown by the following example.

Example 4.1. An I-submaximal space need not be submaximal. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Set $A = \{a, c\}$. Then $\mathrm{Cl}(A) = X$ and $A \notin \tau$. Hence X is not submaximal but I-submaximal space.

Lemma 4.3. *$A \in PIO(X)$ if and only if $A = U \cap D$ for some $U \in \tau$ and τ^*-dense $D \subset X$.*

Proof. If $A \in PIO(X)$, then $A \subset \mathrm{Int}(\mathrm{Cl}^*(A)) = U \in \tau$. Let $D = X - (U - A) = (X - U) \cup A$. Then D is τ^*-dense since $X = \mathrm{Cl}^*(A) \cup (X - \mathrm{Cl}^*(A)) \subset \mathrm{Cl}^*(A) \cup (X - U) = \mathrm{Cl}^*(D)$. Also, $A = U \cap D$. Conversely, if $A = U \cap D$, where $U \in \tau$ and D is τ^*-dense, then $A \subset U$,

$$\mathrm{Int}(\mathrm{Cl}^*(A)) \subset \mathrm{Int}(\mathrm{Cl}^*(U)) \tag{4.1}$$

and $U = U \cap X = U \cap \mathrm{Cl}^*(D) \subset \mathrm{Cl}^*(U \cap D) = \mathrm{Cl}^*(A)$,

$$\mathrm{Int}(\mathrm{Cl}^*(U)) \subset \mathrm{Int}(\mathrm{Cl}^*(A)) \tag{4.2}$$

by (4.1) and (4.2), $\mathrm{Int}(\mathrm{Cl}^*(U)) = \mathrm{Int}(\mathrm{Cl}^*(A))$ so that $A \in PIO(X)$. □

Lemma 4.4. *If (X, τ, I) is I-submaximal then $PIO(X) = \tau$.*

Proof. Clearly $\tau \subset PIO(X)$. Now $A \in PIO(X)$ then $A = U \cap D$ for some $U \in \tau$ and τ^*-dense $D \subset X$. Therefore, if (X, τ, I) is I-submaximal, $D \in \tau$ then $A \in \tau$. □

Definition 4.3. *An ideal topological space (X, τ, I) is said to be P-I-disconnected (briefly P. I .d) if the $\emptyset \neq A^* \in \tau$ for each $A \in \tau$.*

Proposition 4.2. *If a space (X, τ, I) is P-I-disconnected, then $SIO(X) \subset PIO(X)$.*

Proof. Let $A \in SIO(X)$. Then there exists a $U \in \tau$ such that $U \subset A \subset \mathrm{Cl}^*(U)$. Since (X, τ, I) is P-I-disconnected, $\mathrm{Cl}^*(U) \in \tau$ so that $U \subset A \subset \mathrm{Int}(\mathrm{Cl}^*(U))$. This shows that $A \in SIO(X) \subset PIO(X)$. □

I-submaximal space and P-I-disconnected space are independent concepts as the following examples.

Example 4.2. Let (X, τ, I) be the same ideal topological space as at Example 4.1, that is $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Set $A = \{b, c\}$. Then $A^* = \{a, b, d\} \notin \tau$. This shows that X is not P-I-disconnected space by using Definition 4.3 but I-submaximal space.

Example 4.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{b\}\}$. Set $A = \{b\}$. Then $A^* = \emptyset$ and $A \notin \tau$. This shows that X is not I-submaximal by using Theorem 4.8 (c) but P-I-disconnected space.

Theorem 4.9. *Let (X, τ, I) be an I-submaximal and P-I-disconnected space. Then, for a function $f : (X, \tau, I) \to (Y, \nu)$, we have*

(a) *α-I-preirresoluteness $\Leftrightarrow$ pre-I-irresoluteness.*
(b) *α-I-irresoluteness $\Leftrightarrow$ α-pre-I-continuity.*

Proof. This follows from the fact that if (X, τ, I) is an I-submaximal and P-I-disconnected space, then $\tau = \alpha IO(X) = SIO(X) = PIO(X)$. □

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