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Functions with Preclosed Graphs

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Abstract. In this paper the concept of preclosed graphs for functions between topological spaces is introduced with the aid of preopen sets. Some basic properties of functions with a preclosed graph have been obtained.

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1. Introduction

In General Topology basic setting is a mapping from a topological space to another topological space. In this setting determination of continuity condition plays the most important role. Initially functions with closed graph come into existence in the literature of General Topology to ascertain the situation when a function with closed graph is continuous. Quite recently this topic has been extensively studied from different stand points with generalised version of functions with closed graph. In 1969, Long [8] studied the properties of functions with closed graph in great detail. During the last four decades a good deal of effort has been expended in General Topology to extend the notion of continuity with the help of weakened form of open sets or otherwise. Concommitant with the variant forms of continuity, various generalised notions of closed graph have appeared in the literature. Of late, closed graph notion or generalised closed graph notion is no longer used as a tool to ascertain continuity (or generalised continuity) conditions. On the other hand, it is now an active area of research and a large number of topologists have established its far-reaching effect on different concepts of point set topology. In 1983, Dube et al. [3] introduced the notion of semi-closed graph utilising semi-open sets introduced by Levine [7]. The purpose of this note is to define preclosed graph by using preopen sets given in 1982, by Mashhour et al. [9]. In Section 2 of this note some known definitions and results necessary for the presentation of the subject have been listed. Section 3 deals with the definition and basic properties of a preclosed graph.

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2. Preliminaries

Throughout the note (X,τ), (Y,σ) etc. (or simply X, Y etc.) will always denote topological spaces. If A is a subset of the space (X,τ) then the closure (resp. interior) of A in (X,τ) is denoted by $\mathrm{Cl_X(A)}$ (resp. $\mathrm{Int_X(A)}$) or simply by $\mathrm{Cl(A)}$ (resp. $\mathrm{Int(A)}$) if there is no possibility of confusion. For a set $A \subset X$, the family $\{U \in \tau : A \subset U\}$ is denoted by $\Sigma(A)$ and for a point $x \in X$, $\Sigma(x) = \{U \in \tau : x \in U\}$.

We shall require the following known definitions and results.

Definition 2.1. [9] *A subset A of (X,τ) is called preopen (briefly p.o.) if $A \subset \mathrm{Int(Cl(A))}$. The family of all preopen sets is denoted by $PO(X)$ while $PO(X,x)$ denotes the family of p.o. sets containing x.*

Definition 2.2. [9] *$A \subset X$ is called preclosed (briefly pc) if $X - A \in PO(X)$. The family of all preclosed sets is denoted by $PC(X)$.*

Definition 2.3. [2] *For $A \subset X$, the preclosure of A, denoted by $\mathrm{pcl(A)}$ is defined by $\mathrm{pcl(A)} = \cap\{\mathrm{B} : \mathrm{B}$ is preclosed and $\mathrm{B} \supset \mathrm{A}\}$.*

Definition 2.4. [4] *Let $f : (X,\tau) \to (Y,\sigma)$ be any function. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f.*

Definition 2.5. [4] *Let X, Y be topological spaces. A mapping $f : X \to Y$ is said to have a closed graph if its graph $G(f)$ is closed in the product space $X \times Y$.*

Lemma 2.1. [4] *Let $f : X \to Y$ be given. Then $G(f)$ is closed if and only if for each $(x,y) \in (X \times Y) - G(f)$ there exist $U \in \Sigma(x)$ in X and $V \in \Sigma(y)$ in Y such that $f[U] \cap V = \emptyset$.*

Definition 2.6. [6] *A subset $A \subset X$ is said to be preclopen if A is both p.o. and pc.*

Definition 2.7. *A function $f : (X,\tau) \to (Y,\sigma)$ is called*

(i) *precontinuous* [9] *(resp. quasi-precontinuous* [14]*) briefly pc (resp. qpc) if and only if for each $x \in X$ and each $V \in \Sigma(f(x))$ there exists a $U \in PO(X,x)$ such that $f[U] \subset V$ (respectively $f[U] \subset \mathrm{Cl_Y(V)}$);*

(ii) *preirresolute* [16] *briefly pi if and only if $f^{-1}[V] \in PO(X)$ for each $V \in PO(Y)$. (Mashhour et al.* [10] *termed preirresoluteness as M- precontinuity);*

(iii) *p-open* [5] *if and only if $f[A] \in PO(Y)$ for all $A \in PO(X)$.*

Definition 2.8. *X is called*

(i) *pre-T_1* [6] *if and only if for $x, y \in X$ such that $x \neq y$ there exists a p.o. set containing x but not y and a p.o. set containing y but not x;*

(ii) *pre-T_2* [6] *if and only if for $x, y \in X$, $x \neq y$ there exist $U \in PO(X,x)$, $V \in PO(Y,y)$ such that $U \cap V = \emptyset$;*

(iii) *strongly compact* [11] *if every preopen cover of X admits a finite subcover;*

(iv) *pre-regular* [13] *if for each $F \in PC(X)$ and each $x \notin F$ there exist disjoint p.o. sets U, V such that $x \in U$ and $F \subset V$.*

Definition 2.9. [17] *Two subsets A and B of a space X are called preseparated if and only if $A \cap \mathrm{pcl(B)} = \mathrm{pcl(A)} \cap \mathrm{B} = \emptyset$.*

Definition 2.10. [17] *A space X is said to be preconnected if and only if X cannot be expressed as the union of two preseparated sets.*

Lemma 2.2. [11] *Every pc subset of a strongly compact space is strongly compact.*

3. Definition and basic properties of preclosed graphs

Definition 3.1. *For a function $f : X \to Y$, the graph $G(f)$ is said to be preclosed if for each $(x, y) \in (X \times Y) - G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $[U \times V] \cap G(f) = \emptyset$.*

Though the preclosedness of $G(f)$ has been used in literature [15], the authors in [15] gave no formal definition of it.

A useful characterisation of functions with preclosed graph is given below.

Lemma 3.1. *The function $f : X \to Y$ has a preclosed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $f[U] \cap V = \emptyset$.*

Proof. It follows from Definition 3.1 and is omitted. □

Remark 3.1. Evidently every closed graph is preclosed. That the converse is not true is seen from the following example.

Example 3.1. Let $X = \{a, b\}$, $Y = \{a, b, c, d\}$ be two sets endowed with the discrete topology τ and the topology $\sigma = \{\emptyset, \{c, d\}, Y\}$ respectively. Let $f : X \to Y$ be the mapping defined by $f(a) = a$ and $f(b) = b$. Then $G(f)$ is preclosed but not closed.

Remark 3.2. Functions having a preclosed graph need not be pc as shown by

Example 3.2. Let $X = \{a, b, c, d\}$ be equipped with the topology $\tau_1 = \{\emptyset, X, \{c, d\}\}$ and the discrete topology τ_2. Then the graph $G(i)$ of the identity mapping $i : (X, \tau_1) \to (X, \tau_2)$ is preclosed but i is not pc.

Remark 3.3. A pc function need not have a preclosed graph as shown by the following example.

Example 3.3. Let $X = \{a, b, c\}$ be endowed with the discrete topology τ_1 and $\tau_2 = \{\emptyset, X, \{c\}, \{a, c\}\{b, c\}\}$. Then the identity mapping $i : (X, \tau_1) \to (X, \tau_2)$ is pc but $G(i)$ is not preclosed.

Remark 3.4. From Examples 3.2 and 3.3, it is clear that the notions of preclosed graph and pc are independent of each other.

Theorem 3.1. *Let $f : X \to Y$ be pi where X is an arbitrary topological space and Y is pre-T_2. Then $G(f)$ is preclosed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $f(x) \neq y$. Since Y is pre-T_2, there exists $U \in PO(Y, f(x))$, $V \in PO(Y, y)$ such that $U \cap V = \emptyset$. The pi-ness of f implies that $f^{-1}[U] = W \in PO(X, x)$. Hence $f[W] = ff^{-1}[U] \subset U$. It now follows from above that $f[W] \cap V = \emptyset$ which indicates by the Lemma 3.1 that $G(f)$ is preclosed. □

Remark 3.5. The condition pi-ness of f in Theorem 3.1 cannot be replaced by pc as shown by

Example 3.4. Let Y of Example 3.1 be equipped with topologies $\tau_1 = \{\emptyset, Y, \{c\}\}$ and $\tau_2 = \sigma$ of Example 3.1 respectively. Let $i : (Y, \tau_1) \to (Y, \tau_2)$ be the identity map. Then (Y, τ_2) is pre-T_2, i is pc but not pi while $G(i)$ is not preclosed.

Remark 3.6. However, if pre-T_2-ness is replaced by T_2-ness, the pi-ness by qpc then the conclusion of Theorem 3.1 remains true as has been shown by Paul et al. [15].

Theorem 3.2. *Let $f : X \to Y$ be any surjection with $G(f)$ preclosed. Then Y is pre-T_1.*

Proof. Let y_1, $y_2(\neq y_1) \in Y$. The surjectivity of f gives the existence of an element $x_0 \in X$ such that $f(x_0) = y_2$. Now $(x_0, y_1) \in (X \times Y) - G(f)$. The preclosedness of $G(f)$ induces, by Lemma 3.1, $U_1 \in PO(X, x_0)$, $V_1 \in PO(Y, y_1)$ with $f[U_1] \cap V_1 = \emptyset$. Now $x_0 \in U_1 \Rightarrow f(x_0) = y_2 \in f[U_1]$. This and the fact that $f[U_1] \cap V_1 = \emptyset$ guarantee that $y_2 \notin V_1$. Again from the surjectivity of f it follows that there exists a point x_1 such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) - G(f)$ and the preclosedness of $G(f)$ together indicate that there are $U_2 \in PO(X, x_1)$, $V_2 \in PO(Y, y_2)$ with $f[U_2] \cap V_2 = \emptyset$. Now $x_1 \in U_2 \Rightarrow f(x_1) = y_1 \in f[U_2]$ so that $y_1 \notin V_2$. Thus we obtain sets $V_1, V_2 \in PO(Y)$ such that $y_1 \in V_1$ but $y_2 \notin V_1$ while $y_2 \in V_2$ but $y_1 \notin V_2$. Hence Y is pre-T_1. □

Theorem 3.3. *Let $f : X \to Y$ be any p-open surjection with $G(f)$ preclosed. Then Y is pre-T_2.*

Proof. Let y_1, $y_2(\neq y_1) \in Y$. The surjectivity of f gives the existence of a $x \in X$ such that $f(x) = y_2$ which, in its turn, implies that $(x, y_1) \in (X \times Y) - G(f)$. Since $G(f)$ is preclosed, by the Lemma 3.1, one obtains $U \in PO(X, x)$, $V \in PO(Y, y_1)$ with $f[U] \cap V = \emptyset$. The p-openness of f implies that $f[U]$ is p.o. Also $y_2 \in f[U]$. Therefore, there exist $V \in PO(Y, y_1)$ and $f[U] \in PO(Y, y_2)$ such that $f[U] \cap V = \emptyset$. Hence Y is pre-T_2. □

Theorem 3.4. *Let $f : X \to Y$ be injective with $G(f)$ preclosed. Then X is pre-T_1.*

Proof. Let x_1, $x_2(\neq x_1) \in X$. The injectivity of f implies $f(x_1) \neq f(x_2)$ whence one obtains that $(x_1, f(x_2)) \in (X \times Y) - G(f)$. The preclosedness of $G(f)$, by Lemma 3.1, ensures the existence of $U \in PO(X, x_1)$, $V \in PO(Y, f(x_2))$ such that $f[U] \cap V = \emptyset$. Therefore, $f(x_2) \notin f[U]$ and a *fortiori* $x_2 \notin U$. Again $(x_2, f(x_1)) \in (X \times Y) - G(f)$ and preclosedness of $G(f)$, as before gives $A \in PO(X, x_2)$, $B \in PO(Y, f(x_1))$ with $f[A] \cap B = \emptyset$, which guarantees that $f(x_1) \notin f[A]$ and so $x_1 \notin A$. Therefore, we obtain sets U and $A \in PO(X)$ such that $x_1 \in U$ but $x_2 \notin U$ while $x_2 \in A$ but $x_1 \notin A$. Thus X is pre-T_1. □

Corollary 3.1. *Let $f : X \to Y$ be bijective and $G(f)$ be preclosed. Then both X and Y are pre-T_1.*

Proof. It readily follows from Theorems 3.2 and 3.4. □

Theorem 3.5. *For the injective pi $f : X \to Y$ if $G(f)$ is preclosed, then X is pre-T_2.*

Proof. Let x_1, $x_2(\neq x_1) \in X$. From the injectivity of f it follows that $f(x_1) \neq f(x_2)$ and hence $(x_1, f(x_2)) \in (X \times Y) - G(f)$. The preclosedness of $G(f)$ then assures the existence of $U \in PO(X, x_1)$, $V \in PO(Y, f(x_2))$ with $f[U] \cap V = \emptyset$, whence, one obtains $U \cap f^{-1}[V] = \emptyset$. Now the pi-ness of f indicates that $f^{-1}[V] \in PO(X, x_2)$. The disjointness of $U \in PO(X, x_1)$ and $f^{-1}[V] \in PO(X, x_2)$ now yields the pre-T_2-ness of X. □

Corollary 3.2. *If $f : X \to Y$ is bijective, p-open, pi and $G(f)$ is preclosed then both X and Y are pre-T_2.*

Proof. Proof is an immediate consequence of Theorems 3.3 and 3.5. □

To study further properties of preclosed graph we need the following definitions and the lemma.

Definition 3.2. *A function $f : X \to Y$ is said to be preconnected if for every preconnected set U, $f[U]$ is preconnected.*

Definition 3.3. *A mapping $f : X \to Y$ is said to be set preconnected if and only if for every preclopen subset V of $f[X]$, $f^{-1}[V]$ is preclopen in X.*

Definition 3.4. *A topological space X is locally preconnected if for each $x \in X$ and each $U \in PO(X, x)$ there exists a $V \in PO(X, x)$ such that $x \in V \subset U$, where V is preconnected.*

Definition 3.5. *A space X is extremally predisconnected iff the preclosure of every p.o. set is p.o.*

Lemma 3.2. *In a topological space if E be a preconnected set and F be any other set such that $E \subset F \subset \mathrm{pcl(E)}$, then F is preconnected.*

Proof involves standard argument as applied in the classical result and is therefore left out.

Theorem 3.6. *If the injective p-open map $f : X \to Y$ is preconnected and $G(f)$ is preclosed then X is pre-T_2 provided it is T_1 and locally preconnected.*

Proof. Let x_1, $x_2(\neq x_1)$. The injectivity of f induces $f(x_1) \neq f(x_2)$, whence $(x_1, f(x_2)) \in (X \times Y) - G(f)$. From the preclosedness of $G(f)$ one infers that there exist $U_1 \in PO(X, x_1)$, $V \in PO(Y, f(x_2))$ with $f[U_1] \cap V = \emptyset$. Again local preconnectedness of X at x_1, gives the existence of a p.o. preconnected set U such that $x_1 \in U \subset U_1$. From the foregoing it then follows that $f[U] \cap V = \emptyset$. Since f is p-open, $f[U]$ is p.o. We assert that $x_2 \notin \mathrm{pcl(U)}$. Deny it. Then $x_2 \in \mathrm{pcl(U)}$. We shall now show that $U \cup \{x_2\}$ is preconnected. Since X is T_1, $\{x_2\}$ is a closed set. Thus $U \subset U \cup \{x_2\} \subset \mathrm{pcl(U \cup \{x_2\})} = \mathrm{pcl(U)} \cup \{\mathrm{x_2}\} = \mathrm{pcl(U)}$. By Lemma 3.2, $U \cup \{x_2\}$ is then preconnected. Since f is preconnected $f[U \cup \{x_2\}] = f[U] \cup \{f(x_2)\}$ is preconnected which leads to an absurdity because $f[U]$ and V are disjoint p.o. sets and hence they are preseparated. So, $x_2 \notin \mathrm{pcl(U)}$. Setting $U_0 = X - \mathrm{pcl(U)}$ we find $U \in PO(X, x_1)$ and $U_0 \in PO(X, x_2)$ with $U \cap U_0 = \emptyset$. This then implies that X is pre-T_2. □

Theorem 3.7. *Let $f : X \to Y$ be a set preconnected surjection and Y be an extremally predisconnected pre-T_2 space. Then $G(f)$ is preclosed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Now Y being pre-T_2 there is a $H \in PO(Y, y)$ such that $f(x) \notin \mathrm{pcl(H)} = \mathrm{V}$. Since Y is extremally predisconnected, V is preclopen in Y not containing $f(x)$. Again since f is set preconnected surjection $f^{-1}[V]$ is preclopen in X and $x \notin f^{-1}[V]$. Let $U = X - f^{-1}[V]$. Then $U \in PO(X, x)$ and $f[U] \cap V = \emptyset$. Hence $G(f)$ is preclosed. □

Theorem 3.8. *Let X be a space such that $PO(X)$ is a topology. If for the function $f : X \to Y$ where Y is strongly compact, $G(f) \in PC(X \times Y)$, then f is pc.*

Proof. Let $x \in X$, $V \in \Sigma(f(x))$ and $y \in Y - V$. Then $(x, y) \in (X \times Y) - G(f)$. So there exist $U_y \in PO(X, x)$, $V_y \in PO(Y, y)$ such that

$$f[U_y] \cap V_y = \emptyset. \tag{3.1}$$

This relation holds for every $y \in Y - V$. Clearly $\mathcal{V} = \{V_y : y \in Y - V\}$ is a cover of $Y - V$ by p.o. sets. Now Y is precompact and $Y - V$ is preclosed. Hence, by Lemma 2.2, $Y - V$ is precompact. So $\mathcal{V}$ has a finite subfamily $\{V : i = 1, 2, \ldots, n\}$ such that

$$Y - V \subset \cup_{i=1}^{n} V_{y_i}.$$

Let $\{U_{y_i} : i = 1, 2, \ldots, n\}$ be the corresponding sets of $PO(X, x)$ satisfying the relation of type (3.1). Set $U = \cap_{i=1}^{n} U_{y_i}$. Since X enjoys the property P, $U \in PO(X, x)$. If $\alpha \in U$, then $f(\alpha) \notin V_{y_i}$ for all $i = 1, 2, \ldots, n$. This implies that $f(\alpha) \notin Y - V$, so that $f(\alpha) \in V$. Since α is arbitrary it follows that $f[U] \subset V$ which guarantees the pc of f. □

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