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A Subcritical Flow Over a Stepping Bottom

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Abstract. A free-surface flow on a channel with step bottom is solved numerically by an integral equation method. The flow is assumed to be steady, 2 dimensional and irrotational, meanwhile the fluid is inviscid and incompressible, and the effect of gravity is not negligible. The numerical result show that solutions with a train of waves are obtained for subcritical flow, and for various height of the step. The waves are characterized as long wave with ratio amplitude to wave-length of order 10^{-2} or smaller.

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1. Introducton

This paper is concerned with the numerical calculation of 2-D fluid flow in an open channel. The flow is uniform far upstream and is disturbed by an obstruction on the bottom of the channel, which is a sloping step. The parameters corresponding to the incoming flow and the geometry of the step affect the profile of the flow after it passes the step. However, we only consider subcritical flow, i.e. the velocity of the incoming flow U is less than $\sqrt{gD}$, where g is the acceleration of gravity and D is the uniform depth. We expect that this incoming flow generates a train of waves behind the step, as has been obtained by some researchers for different obstruction shapes.

Forbes and Schwartz [3] and Zhang and Zhu [7] solved the similar problem for the uniform flow passing a semi-circular obstruction, and a solution with a train of waves was obtained for subcritical flow. Solutions without waves may be seen in Forbes [2], and Dias and Vanden-Broeck [1] who solved the free-surface flow passing a triangle obstruction with both flows, upstream and downstream, subcritical or supercritical ($U > \sqrt{gD}$).

In this paper the free-surface flow is solved numerically by a boundary element method. An integral equation is constructed from the related boundary value problem by firstly introducing a hodograph variable, and then transforming the flow

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domain into an artificial one which is a half complex plane. The integral equation expresses the hodograph variable on the boundary of the artificial plane. Numerically, this equation can be solved by discretising the domain of integration to construct a system of nonlinear equations, and the Newton iteration method is applied. This boundary element method has been successfully applied to some problems of nonlinear free-surface flow. This can be seen, for example, in Wiryanto [4, 5] and Wiryanto and Tuck [6]. The numerical solution is then used to observe the characteristics of the generated waves behind the step by calculating the ratio of the amplitude to the wave-length for various related parameters, i.e. the uniform flow in $U/\sqrt{gD}$ and the slope and height of the step.

In presenting this paper, we organize the sections as follows. In section 2 we formulate the problem into the integral equation as described above. This is followed by presenting the numerical procedure in solving the integral equation that is given in section 3. The result of the calculations is then discussed in section 4.

2. Mathematical formulation

A steady, irrotational 2-D flow of an ideal fluid is considered. Far upstream the fluid flows uniformly with velocity U and depth D. This flow is disturbed by the existence of a sloping step obstruction on the bottom of the channel, illustrated in Figure 1. The system of coordinates is chosen Cartesian with the horizontal x-axis on the lowest bottom of the channel and the vertical y-axis passing the corner point A of the sloping step. Therefore, the step makes angle α and height H to the horizontal axis. The flow domain is then expressed in complex variable $z = x + iy$ with the complex potential $f = \phi + i\psi$, where ϕ and ψ represent respectively the potential function and the stream-function. In the f-plane the flow domain is an infinite strip of height UD with the bottom boundary corresponding to the bottom topography of the channel and the top boundary corresponding to the surface of the fluid. As the reference of the coordinate, we choose the center corresponding to the point A in the physical plane.

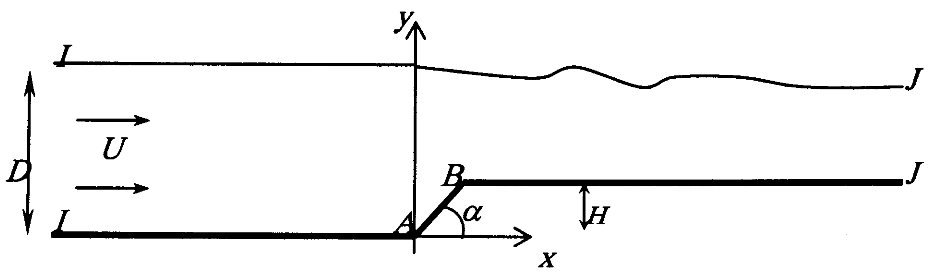

Figure 1. Sketch of the flow in physical plane $z = x + iy$.

Mathematically, the problem is to determine the complex potential $f(z)$ satisfying Laplace equation ($\nabla^2 f = 0$) in the flow domain, followed by kinematic and dynamic conditions. The first condition represents no flow crossing the solid and

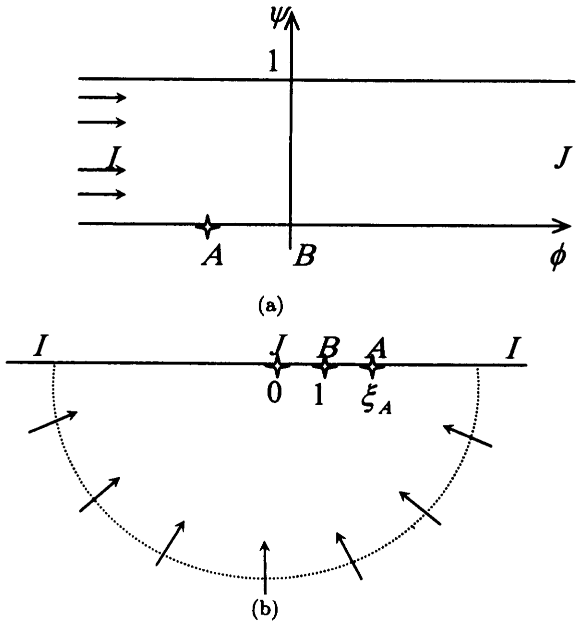

Figure 2. Sketch of the flow domain in (a) f-plane (b) ζ-plane.

free boundaries, and the second condition represents the fact that the pressure is hydrostatic along the free boundary. In terms of the potential function our task is to solve

(2.1) $$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the flow domain}$$

subject to

(2.2a) $$\phi_x = 0 \text{ on the bottom topography } y = b(x),$$

(2.2b) $$\phi_x y_x = \phi_y \text{ on the free boundary } y = y(x),$$

(2.2c) $$\frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + gy = \frac{1}{2}U^2 + gD \text{ on } y = y(x)$$

The constant on the right hand side of (2.2c) is obtained from the upstream uniform flow.

For convenience, the problem is non-dimensionalized by defining D and U as the reference for the length and velocity respectively. This changes the above formula only to (2.2c) becoming

(3.3) $$\frac{F^2}{2}\left(\phi_x^2 + \phi_y^2\right) + y = \frac{F^2}{2} + 1$$

where $F = U/\sqrt{gD}$ is the upstream Froude number, and the height of the step is then denoted $h = H/D$. Meanwhile, the flow domain in the f-plane changes to an infinite strip of height 1.

The next step is to formulate the problem into an integral equation. To do so we first map the flow domain of the f-plane to a half lower plane by

$$f = -\frac{1}{\pi}\log\zeta \tag{2.4}$$

where $\zeta = \xi + i\eta$ is the complex variable in the artificial plane. The relation (2.4) maps the free boundary, $\psi = 1$, $-\infty < \phi < \infty$, to the half real axis of ζ, i.e. $\eta = 0, -\infty < \xi < 0$; and the solid boundary $\psi = 0$, $-\infty < \phi < \infty$, to the other half real axis. Points A and B in the f-plane are mapped to $\zeta = \xi_A$ and $\zeta = 1$ respectively. The sketch of the flow domain in both planes is shown in Figure 2.

Next, we introduce the hodograph variable $\Omega = \tau - i\theta$ relating to the complex velocity

$$\frac{df}{dz} = e^{\Omega}. \tag{2.5}$$

The components τ and θ represent the modulus (in logarithm) and argument of the velocity vector on streamlines. Since the solid boundary is also a streamline, the kinematic condition along this boundary is expressed more simply in this hodograph variable, i.e.

$$\theta = \begin{cases} 0, & 0 < \xi < 1 \cup \xi > \xi_A \\ \alpha, & 1 < \xi < \xi_A. \end{cases} \tag{2.6}$$

Similarly, the dynamic condition (2.3) can be expressed as

$$F^2 e^{2\tau} + 2y = F^2 + 2 \tag{7}$$

along the free boundary $\xi < 0$. Equation (2.7) is the integral equation for θ after substituting the values of y and τ, which are calculated from

$$\frac{dy}{d\xi} = \frac{-e^{-\tau}}{\pi\xi}\sin\theta \tag{2.8}$$

and

$$\tau(\xi) = \frac{\alpha}{\pi}\ln\left|\frac{\xi_A - \xi}{1-\xi}\right| + \frac{1}{\pi}PV\int_{-\infty}^{0}\frac{\theta(s)}{s-\xi}ds. \tag{2.9}$$

Note that the integral in (9) is Principle Cauchy Value, denoted by "PV".

The relation (2.8) is the imaginary part of $dz/d\xi$, where this differentiation can be expressed in hodograph variable Ω as

$$\frac{dz}{d\xi} = \frac{e^{-\Omega}}{\pi\xi}. \tag{2.10}$$

This is the result of applying (2.4) and (2.5) to the chain rules for $dz/d\xi$. Meanwhile, (2.9) is obtained from the Cauchy theorem applied to Ω along a closed path consisting of the real axes ξ, a lower semi circle $|\zeta| = \infty$, and a small circular path around a point ζ. For $\mathrm{Im}(\zeta) < 0$, the Cauchy theorem gives

$$\Omega(\zeta) = -\frac{1}{2\pi i}PV\int_{-\infty}^{\infty}\frac{\Omega(s)}{s-\zeta}ds, \tag{2.11}$$

since Ω as $|\zeta| \to \infty$. Now let $\text{Im}(\zeta) \to 0^-$, and the real part of (2.11) tends to

$$\tau(\xi) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\theta(s)}{s - \xi} ds. \tag{2.12}$$

Finally the relation (2.9) is obtained from (2.12) by substituting the value of θ along the solid boundary from (2.6).

3. Numerical procedure

From the formulation above, the problem is to determine $\theta(\xi)$ along the free boundary or in the interval $-\infty < \xi < 0$, satisfying the integral equation (2.7). In solving the problem numerically, we first truncate the domain of integration in (2.9) to a finite interval $[-T, -R]$, where T and R are relatively a big number and a small one, representing the flow far upstream and downstream respectively. This integral is then approximated by the trapezoidal method. To do so, we discretize the interval $[-T, -R]$ into N subintervals having the same width in terms of variable ϕ, to get better accuracy in the numerical calculation. The discrete points are

$$\xi_j = -e^{\pi \phi_j} \tag{2.13}$$

where $\phi_j = -\phi_0 + j\frac{2\phi_0}{N}$, $j = 0, 1, \ldots, N$; and $T = e^{\pi\phi_0}$, $R = e^{-\pi\phi_0}$. We then denote $\theta_j = \theta(\xi_j)$ as the unknowns, except θ_0 defined to be 0, representing the fact that the flow is uniform.

The next step is to construct a system of non-linear equations from (2.7). This requires N collocation points ξ_j^*, with each chosen in the interval between ξ_{j-1} and ξ_j. We define those points corresponding to the mid points $\phi_j^* = -\phi_0 + (j - .5)\frac{2\phi_0}{N}$, between ϕ_j and ϕ_{j-1}, related as given in (2.13). Meanwhile the value of θ at ξ_j^* is denoted by θ_j^* as the average between θ_j and θ_{j-1}. Therefore, for each ξ_j^* Equation (2.6) gives one non-linear equation, after substituting $\tau(\xi_j^*)$ and $y(\xi_j^*)$. The value of the first function is calculated from

$$\tau(\xi_j^*) \approx \frac{\alpha}{\pi} \ln\left|\frac{\xi_A - \xi_j^*}{1 - \xi_j^*}\right| + \frac{1}{2\pi} \sum_{i=0}^{N-1} \left(\frac{\theta_{i+1}}{\xi_{i+1} - \xi_j^*} - \frac{\theta_i}{\xi_i - \xi_j^*} \right) (\xi_{i+1} - \xi_i) \tag{2.14}$$

as the approximation of (2.9), and the second function $y(\xi_j^*)$ is obtained by integrating (2.8) and approximating similar to (2.14).

The above description gives N equations from the collocation points to calculate N unknowns $\theta_1, \theta_2, \ldots, \theta_N$ for given F, ξ_A, α. This discretized form can be solved numerically by Newton's method. The input ξ_A is used to replace the physical parameter h, and h can be evaluated afterward. On the other hand, giving h as an input will increase the number of unknown in the system of equations, and then an extra equation can be constructed from the relation between h and ξ_A. Consequently, the predicted value of ξ_A in each iteration of the Newton's method will change, followed by the changing of the discretizing interval $[1, \xi_A]$ due to the extra equation. Therefore, this makes it more difficult to guarantee the convergence of the Newton iteration process.

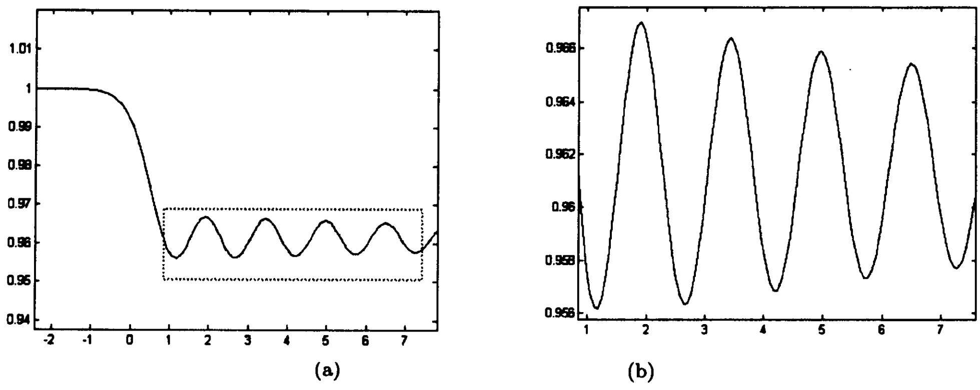

Figure 3. Plot of the surface for $F = 0.4$, $h = 0.16$. (a) The surface before and after passing the step. (b) Zoom of the surface indicated in Figure 3a.

4. Numerical results

The numerical procedure described above is used to solve the free surface problem for the case of subcritical flow ($F < 1$). The result of the calculation is discussed here based on the profile of the free surface, where a train of waves appears on the free surface behind the step, followed by a drop in the mean free surface height. This profile is observed for various values of F, h and α.

In presenting the results, most of our computations were performed with $N = 100$ for discretizing the domain of integration $[-10^4, -10^{-10}]$, and we enforced convergence of Newton's method to within error 10^{-8}. These values give sufficient good accuracy to the output which is performed to the height of the step h, since the result of the Newton iteration is then used to calculate this quantity which corresponds to the input ξ_A. We found that the calculation of is about 3-figure accuracy for discretizing the interval $[1, \xi_A]$ into 100 subintervals.

Figure 3 shows a plot of the surface for $F = 0.4, h = 0.14$ with the slope of the step $\alpha = 21°$. In Figure 3a, the surface is shown from upstream to relatively far from the step, and the indicated surface, where the train of waves appears, is shown in Figure 3b. The plot shows that the surface forms waves with smaller amplitude for farther down from the step. Meanwhile, the distance between the crest and trough increases. As a comparison, the amplitude for the first wave and the last wave is 0.011 and 0.008, and the wave-length changes from 1.463 to 1.528.

Figure 4 shows a plot of two surfaces as the result of calculation using two different Froude numbers, namely $F = 0.4$ and 0.33, and $h = 0.173$, $\alpha = 21°$. The numerical procedure produces surfaces with different waves. For larger the train of waves has larger amplitude and larger wave-length. This characteristic can be compared by mean of a quantity such as the ratio between the amplitude and the wave-length a/λ. This quantity represents a train of waves on the surface by taking the biggest value of a/λ. This is obtained for the nearest wave to the step. Hence, our calculations give $a/\lambda = 0.012$ for $F = 0.4$, and $a/\lambda = 0.003$ for $F = 0.33$.

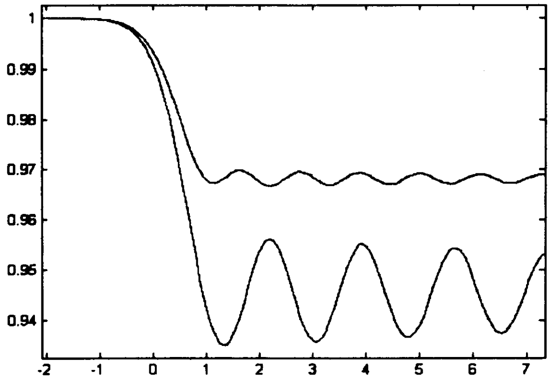

Figure 4. Plot of two surfaces for different Froude number $F0.4$ (large waves) and $F = 0.33$ (small waves), $h = 0.173$.

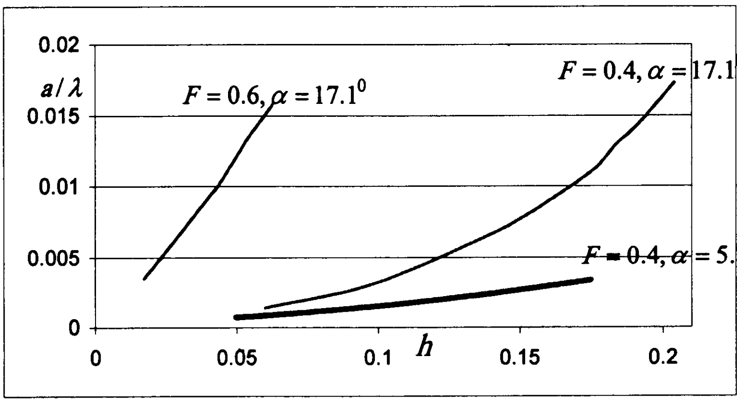

Figure 5. Plot of h versus a/λ for different values F and α.

The effect of the step height on the appearing waves can be seen by comparing plots in Figure 3 and Figure 4 for the same value $F = 0.4$. According to the quantity a/λ, we found that increasing is followed by increasing . This quantity is then used to represent some calculations, instead of showing some surface plots. We show in Figure 5 plot of h versus a/λ for different values of F and α as indicated near the curves. In general, the obstruction on the subcritical flow generates long waves in which the wave-length can reaches 100 times the amplitude, or larger.

5. Conclusion

We have presented the numerical calculation of subcritical flow passing a sloping step on the bottom of a channel, using an integral equation. Our calculations show that the fluid behind the obstruction is shallower than in the front, and the obstruction generates waves. From some values of related parameters we found that the waves are typically long wave, with ratio amplitude to wave-length of order 10^{-2} or smaller.

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