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Structures of Fuzzy Ideals of Γ-Ring

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Abstract. In this paper we define some compositions of fuzzy ideals in a Γ-ring and study the structures of the set of fuzzy ideals of a Γ-ring. Also we characterize Γ-field, Noetherian Γ-ring, etc. with the help of fuzzy ideals via operator rings of Γ-rings.

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1. Introduction

The notion of fuzzy ideals in a Γ-ring was introduced by Jun and Lee in [6]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Later in [5] Jun and Hong defined normalized fuzzy ideals and fuzzy maximal ideals in Γ-rings and studied them. In Section 3 of this paper we define some compositions of fuzzy ideals of a Γ-ring and study the structures of the set of fuzzy ideals of a Γ-ring. We show that $FLI(M)$, the set of all fuzzy left ideals of a Γ-ring M, is a zerosumfree hemiring having infinite element 1, under the operations of sum and composition of fuzzy left ideals. Similar results hold for the set of fuzzy right ideals and that of fuzzy ideals of M. In Section 4 we define a correspondence between the set of all fuzzy ideals of a Γ-ring and the set of all fuzzy ideals of the operator rings of the Γ-ring. We obtain that the lattice of all left (resp. right, two sided) fuzzy ideals is isomorphic to the lattice of all left (resp. right, two sided) fuzzy ideals of the operator ring of the Γ-ring. Using these results we characterize Γ-field, Noetherian Γ-ring etc.

2. Preliminaries

Definition 2.1. [1] *Let M and Γ be two additive abelian groups. M is called a Γ-ring if the following conditions are satisfied for all $a, b, c \in M$ and for all $\alpha, \beta, \gamma \in \Gamma$:*

(i) $a\alpha b \in M$,
(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ *and*
(iii) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

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Definition 2.2. [9] *A subset A of M is called a left (resp. right) ideal of M if A is an additive subgroup of M and $m\alpha a \in A$ (resp. $a\alpha m \in A$) for all $m \in M$, $\alpha \in \Gamma$ and $a \in A$. If A is both a left and a right ideal of M, then A is called a two sided ideal of M or simply an ideal of M.*

Definition 2.3. [9] *Let M be a Γ-ring and F the free abelian group generated by $\Gamma \times M$. Then $A = \{\sum_i n_i(\gamma_i, x_i) \in F : a \in M \Rightarrow \sum n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$, the factor group of F by A. Let us denote the coset $(\gamma, x)+A$ by $[\gamma, x]$. It can be verified that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$, and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in R by $\sum_i[\alpha_i, x_i]\sum_j[\beta_j, y_j] = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$. Then R forms a ring. If we define composition on $M \times R$ into M by $a\sum_i[\alpha_i, x_i] = \sum_i a\alpha_i x_i$ for $a \in M$, $\sum_i[\alpha_i, x_i] \in R$, then M is a right R-module, and we call R the right operator ring of the Γ-ring M. Similarly, we can construct a left operator ring L of M so that M is a left L-module. For subsets $N \subseteq M$, $\phi \subseteq \Gamma$, we denote by $[\phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R, where $\gamma_i \in \phi$, $x_i \in N$ and we denote by $[(\Phi, N)]$ the set of all elements $[\phi, x]$ in R where $\phi \in \Phi$, $x \in N$. Thus in particular, $R = [\Gamma, M]$ and $L = [M, \Gamma]$. If there exists an element $\sum_i[\delta_i, e_i] \in R$ such that $\sum_i x\delta_i e_i = x$ for every element x of M, then it is called right unity of M. It can be verified that $\sum_i[\delta_i, e_i]$ is the unity of R. Similarly we can define the left unity $\sum_j[f_j, \gamma_j]$ which is the unity of the left operator ring L.*

Definition 2.4. [6] *A nonempty fuzzy subset μ (i.e., $\mu(x) \neq 0$ for some $x \in M$) of a Γ-ring M is called a fuzzy left(resp. right) ideal of M if, (i) $\mu(x-y) \geq \mu(x)\wedge\mu(y)$, (ii) $\mu(x\alpha y) \geq \mu(y)$ (resp. $\mu(x\alpha y) \geq \mu(x)$) for all $x, y \in M$, and all $\alpha \in \Gamma$.*

Definition 2.5. *A Γ-ring M is said to be commutative if $a\gamma b = b\gamma a$ for all $a, b \in M$ and for all $\gamma \in \Gamma$.*

Definition 2.6. [3] *A commutative Γ-ring M is called a Γ-field if for every non-zero element a of M and for every pair of nonzero elements $\gamma_1, \gamma_2 \in \Gamma$, there exists an element a' in M such that $a\gamma_1 a'\gamma_2 b = b$ for all $b \in M$.*

Definition 2.7. [4] *A hemiring [resp. semiring] is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:*

(1) *$(R, +)$ is a commutative monoid with identity element 0;*
(2) *$(R, .)$ is a semigroup [resp. monoid with identity element 1_R];*
(3) *Multiplication distributes over addition from either side;*
(4) *$0r = 0 = r0$ for all $r \in R$;*
(5) *$1_R \neq 0$.*

A hemiring R is said to be zerosumfree iff $r + r' = 0$ implies that $r = r' = 0$ for all $r, r' \in R$. An element a of a hemiring R is infinite iff $a + r = a$ for all $r \in R$.

3. Operations on fuzzy ideals

Throughout this paper M denotes a Γ-ring with left unity and right unity and $FLI(M)$ (resp. $FRI(M)$, $FI(M)$) denotes the set of all fuzzy left ideals (resp. fuzzy right ideals, fuzzy ideals) of M. Also we assume that for any fuzzy left (resp. right, two sided) ideal σ of M, $\sigma(0_M) = 1$.

Definition 3.1. *Let μ, σ be two fuzzy subsets of M. Then the sum $\mu \oplus \sigma$, product $\mu \, \Gamma \, \sigma$ and composition $\mu \circ \sigma$ of μ and σ are defined as follows:*

$$
\begin{aligned}
(\mu \oplus \sigma)(x) &= \begin{cases} \sup_{x=u+v}[\min[\mu(u), \sigma(v)]] & \text{for } u, v \in M \\ 0 & \text{otherwise.} \end{cases} \\
(\mu \, \Gamma \, \sigma)(x) &= \begin{cases} \sup_{x=u\gamma v}[\min[\mu(u), \sigma(v)]] & \text{for } u, v \in M \text{ and } \gamma \in \Gamma \\ 0 & \text{otherwise.} \end{cases} \\
(\mu \circ \sigma)(x) &= \begin{cases} \sup[\min_i[\min[\mu(u_i), \sigma(v_i)]]], & 1 \le i \le n, x = \sum_{i=1}^{n} u_i \gamma_i v_i, \\ & u_i, v_i \in M, \gamma_i \in \Gamma \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Proposition 3.1. *Let μ, σ be two fuzzy ideals of M. Then $\mu \, \Gamma \, \sigma \subseteq \mu \circ \sigma \subseteq \mu \cap \sigma$.*

Proof. From the definitions of $\mu \, \Gamma \, \sigma$ and $\mu_\circ \sigma$, it follows that $\mu \, \Gamma \, \sigma \subseteq \mu_\circ \sigma$. Let $x \in M$ and $x = \sum_{i=1}^{n} u_i \gamma_i v_i$, u_i, $v_i \in M$, $\gamma_i \in \Gamma$ for $i = 1, 2, \ldots, n$. Now

$$
\begin{aligned}
\mu(x) &= \mu(\sum_{i=1}^{n} u_i \gamma_i v_i) \\
&\ge \min\{\mu(u_1 \gamma_1 v_1), \mu(u_2 \gamma_2 v_2), \ldots, \mu(u_n \gamma_n v_n)\} \\
&\ge \min\{\mu(u_1), \mu(u_2), \ldots, \mu(u_n)\}.
\end{aligned}
$$

Similarly

$$\sigma(x) \ge \min\{\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_n)\}.$$

Thus

$$(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} \ge \min_i[\min[\mu(u_i), \sigma(v_i)]].$$

So $(\mu \cap \sigma)(x) \ge \sup[\min_i[\min[\mu(u_i), \sigma(v_i)]]]$, $1 \le i \le n$, $x = \sum_{i=1}^{n} u_i \gamma_i v_i$, $u_i, v_i \in M$, $\gamma_i \in \Gamma = (\mu \circ \sigma)(x)$. Also if $\mu \circ \sigma(x) = 0$, then $\mu \circ \sigma(x) \le \mu \cap \sigma(x)$. So $\mu \circ \sigma \subseteq \mu \cap \sigma$. Thus $\mu \, \Gamma \, \sigma \subseteq \mu \circ \sigma \subseteq \mu \cap \sigma$. □

Proposition 3.2. *Let μ_1, $\mu_2 \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then $\mu_1 \oplus \mu_2 \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$].*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Also let $(\mu_1 \oplus \mu_2)(y) > (\mu_1 \oplus \mu_2)(x)$. Then there exist $p, q \in M$ such that $y = p + q$ and for any $u, v \in M$, for which $x = u + v$, $\min[\mu_1(p), \mu_2(q)] > \min[\mu_1(u), \mu_2(v)]$. Let $u, v \in M$ be such that $x = u + v$. Now $x - y = (u - p) + (v - q)$. So

$$
\begin{aligned}
(\mu_1 \oplus \mu_2)(x - y) &\ge \min[\mu_1(u - p), \mu_2(v - q)] \\
&\ge \min[\min[\mu_1(u), \mu_1(p)], \min[\mu_2(v), \mu_2(q)]] \\
&= \min[\min[\mu_1(u), \mu_2(v)], \min[\mu_1(p), \mu_2(q)]] \\
&= \min[\mu_1(u), \mu_2(v)].
\end{aligned}
$$

So

$$\begin{aligned}(\mu_1 \oplus \mu_2)(x-y) &\geq \sup_{x=u+v}[\min[\mu_1(u), \mu_2(v)]], u, v \in M \\ &= (\mu_1 \oplus \mu_2)(x) \\ &= \min[(\mu_1 \oplus \mu_2)(x), (\mu_1 \oplus \mu_2)(y)].\end{aligned}$$

Similarly we can show that $(\mu_1 \oplus \mu_2)(x-y) \geq \min[(\mu_1 \oplus \mu_2)(x), (\mu_1 \oplus \mu_2)(y)]$, in all other cases. Again, let $y = p+q$, $p, q \in M$. Then $x\gamma y = x\gamma p + x\gamma q$, $x \in M$ and $\gamma \in \Gamma$. Now $(\mu_1 \oplus \mu_2)(x\gamma y) \geq \min[\mu_1(x\gamma p), \mu_2(x\gamma q)] \geq \min[\mu_1(p), \mu_2(q)]$. Thus $(\mu_1 \oplus \mu_2)(x\gamma y) \geq \sup_{y=p+q}[\min[\mu_1(p), \mu_2(q)]]$, $p, q \in M = (\mu_1 \oplus \mu_2)(y)$. Lastly, since $\mu_1(0_M) = \mu_2(0_M) = 1$, $(\mu_1 \oplus \mu_2)(0_M) = 1$. So $\mu_1 \oplus \mu_2 \in FLI(M)$. □

Proposition 3.3. *Let $\mu, \sigma, \delta \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then*

(i) $\mu \oplus \sigma = \sigma \oplus \mu$,
(ii) $(\mu \oplus \sigma) \oplus \delta = \mu \oplus (\sigma \oplus \delta)$,
(iii) $\mu \subseteq \mu \oplus \sigma$,
(iv) *if* $\mu \subseteq \sigma$, *then* $\mu \oplus \delta \subseteq \sigma \oplus \delta$,
(v) $\mu \oplus \mu = \mu$,
(vi) $\theta \oplus \mu = \mu = \mu \oplus \theta$ *where* $\theta(\in FLI(M))$ *is defined by*

$$\theta(x) = \begin{cases} 1 & \text{if } x = 0_M, x \in M \\ 0 & \text{if } x \neq 0_M. \end{cases}$$

Proof. The proof is a routine matter of verification and so we omit it. □

Proposition 3.4. *Let $\mu, \sigma \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then $\mu \circ \sigma \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$].*

Proof. The proof is similar to the proof of the Proposition 3.2 and so we omit it. □

Proposition 3.5. *Let $\mu, \sigma, \delta \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then $\mu \Gamma \sigma \subseteq \delta$ iff $\mu \circ \sigma \subseteq \delta$.*

Proof. If $\mu \circ \sigma \subseteq \delta$, then $\mu \Gamma \sigma \subseteq \mu \circ \sigma \subseteq \delta$. Conversely, let $\mu \Gamma \sigma \subseteq \delta$. Let $x \in M$ be such that $x = \sum_{i=1}^{n} u_i \gamma_i v_i, u_i, v_i \in M$, $\gamma_i \in \Gamma$ for $1 \leq i \leq n$. Now

$$\begin{aligned}\delta(x) &= \delta(\sum_{i=1}^{n} u_i \gamma_i v_i) \\ &\geq \min[\delta(u_1\gamma_1 v_1), \delta(u_2\gamma_2 v_2), \ldots, \delta(u_n\gamma_n v_n)] \\ &\geq \min[(\mu \Gamma \sigma)(u_1\gamma_1 v_1), (\mu \Gamma \sigma)(u_2\gamma_2 v_2), \ldots, (\mu \Gamma \sigma)(u_n\gamma_n v_n)] \\ &\geq \min[\min[\mu(u_1), \sigma(v_1)], \ldots, \min[\mu(u_n), \sigma(v_n)]].\end{aligned}$$

So

$$\delta(x) \geq \sup_{x=\sum_{i=1}^{n} u_i\gamma_i v_i} [\min_i[\min[\mu(u_i), \mu(v_i)]]] = (\mu \circ \sigma)(x).$$

Also if $(\mu_\circ \sigma)(x) = 0$, then $\mu \circ \sigma(x) \leq \delta(x)$. Thus $\mu \circ \sigma \subseteq \delta$.

□

Proposition 3.6. *Let $\mu, \sigma, \delta \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then*

(i) *if $\mu \subseteq \sigma$, then $\mu \circ \delta \subseteq \sigma \circ \delta$,*
(ii) $(\mu \circ \sigma) \circ \delta = \mu \circ (\sigma \circ \delta)$,
(iii) $\mu \circ \sigma = \sigma \circ \mu$ *if M is commutative,*
(iv) $1 \circ \mu = \mu$ *where $1 \in FLI(M)$ is defined by $1(x) = 1$ for all $x \in M$ [resp. $\mu \circ 1 = \mu$, $1 \circ \mu = \mu \circ 1 = \mu$].*

Proof. The proof of (i) to (iii) follows from the definitions of compositions of fuzzy ideals and so we omit it. (iv) As M is with left unity $\sum_j [f_j, \gamma_j] \in L$ which is defined by $\sum_j f_j \gamma_j x = x$ for every element x in M, it follows form definition that $1_\circ \mu = \mu$. □

Similarly we can prove the following proposition.

Proposition 3.7. *Let $\mu, \sigma, \delta \in FLI(M)$ [resp. $FRI(M)$, $FI(M)$]. Then*

(i) $\mu \circ (\sigma \oplus \delta) = \mu \circ \sigma \oplus \mu \circ \delta$,
(ii) $(\sigma \oplus \delta) \circ \mu = \sigma \circ \mu \oplus \delta \circ \mu$.

Theorem 3.1. *Let M be a Γ-ring. Then $FLI(M)$ [resp. $FRI(M)$, $FI(M)$] is a zerosumfree hemiring(resp. hemiring, semiring) having infinite element 1 under the operations of sum and composition of fuzzy left ideals.*

Proof. From the Propositions 3.2, 3.3, 3.4, 3.6 and 3.7, it follows that $FLI(M)$ is a hemiring under the operations of sum and composition of fuzzy left ideals. Now $(1 \oplus \mu)(x) = \sup_{x=u+v}[\min[1(u), \mu(v)]] \geq \min[1(x), \mu(0_M)] = 1(x) \geq (1 \oplus \mu)(x)$ for all $x \in M$. So $1 \oplus \mu = 1$ for all $\mu \in FLI(M)$. Thus 1 is an infinite element of $FLI(M)$. Lastly we assume that $\mu \oplus \sigma = \theta$ for $\mu, \sigma \in FLI(M)$. Then $\mu \subseteq \mu \oplus \sigma = \theta \subseteq \mu$. So $\mu = \theta$. So $FLI(M)$ is zerosumfree. Hence the theorem. □

Lemma 3.1. [6] *Intersection of a nonempty collection of fuzzy left ideals (resp. fuzzy right ideals, fuzzy ideals) is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of M.*

Theorem 3.2. *$FLI(M)$ [resp. $FRI(M)$, $FI(M)$] is a complete lattice.*

Proof. We define a relation '$\leq$' on $FLI(M)$ as follows $\mu_1 \leq \mu_2$ iff $\mu_1(x) \leq \mu_2(x)$ for all $x \in M$. Then $FLI(M)$ is a poset w.r.t. '$\leq$'. Now $1 \in FLI(M)$ and $\mu \leq 1$ for all $\mu \in FLI(M)$. So 1 is the greatest element of $FLI(M)$. Let $\{\mu_i, i \in I\}$ be a nonempty family of fuzzy left ideals of M. Then by Lemma 3.1, it follows that $\cap_{i \in I} \mu_i \in FLI(M)$. Also it is the glb of $\{\mu_i | i \in I\}$. Consequently $FLI(M)$ is a complete lattice. □

4. Corresponding fuzzy ideals

Throughout this paper R denotes the right operator ring and L denotes the left operator ring of M.

Definition 4.1. *For a fuzzy subset μ of R, we define a fuzzy subset μ^* of M by $\mu^*(a) = \inf_{\gamma \in \Gamma} \mu([\gamma, a])$ where $a \in M$. For a fuzzy subset σ of M, we define a fuzzy subset $\sigma^{*'}$ of R by $\sigma^{*'}(\sum_i [\alpha_i, a_i]) = \inf_{m \in M} \sigma(\sum_i m \alpha_i a_i)$ where $\sum_i [\alpha_i, a_i] \in R$.*

Definition 4.2. *For a fuzzy subset δ of L, we define a fuzzy subset δ^+ of M by $\delta^+(a) = \inf_{\gamma\in\Gamma} \delta([a,\gamma])$, where $a \in M$. For a fuzzy subset η of M, we define a fuzzy subset $\eta^{+'}$ of L by $\eta^{+'}(\sum\limits_i [a_i, \alpha_i]) = \inf_{m\in M} \eta(\sum\limits_i a_i\alpha_i m)$ where $\sum\limits_i [a_i, \alpha_i] \in L$.*

Lemma 4.1. *If $\{\mu_i | i \in I\}$ is a collection of fuzzy subsets of R, then*

$$\cap_{i\in I}\mu_i^* = (\cap_{i\in I}\mu_i)^*.$$

Proof. Let $x \in M$. Now

$$\begin{aligned}
(\cap_{i\in I}\mu_i)^*(x) &= \inf_{\gamma\in\Gamma}[(\cap_{i\in I}\mu_i)([\gamma, x])] \\
&= \inf_{\gamma\in\Gamma}[\inf_{i\in I}(\mu_i[\gamma, x])] \\
&= \inf_{i\in I}[\inf_{\gamma\in\Gamma}[\mu_i([\gamma, x])]] \\
&= \inf_{i\in I}[\mu_i^*(x)] \\
&= (\cap_{i\in I}\mu_i^*)(x).
\end{aligned}$$

So $\cap_{i\in I}\mu_i^* = (\cap_{i\in I}\mu_i)^*$. □

Proposition 4.1. *If $\mu \in FI(R)$ [resp. $FRI(R)$, $FLI(R)$], then $\mu^* \in FI(M)$ [resp. $FRI(M)$, $FLI(M)$].*

Proof. Let μ be a fuzzy ideal of R. Then $\mu(0_R) = 1$. Now

$$\mu^*(0_M) = \inf_{\gamma\in\Gamma} \mu([\gamma, 0_M]) = \inf_{\gamma\in\Gamma} \mu(0_R) = 1.$$

So μ^* is nonempty. Let $a, b \in M$ and $\alpha \in \Gamma$. Now

$$\begin{aligned}
\mu^*(a-b) &= \inf_{\gamma\in\Gamma} \mu([\gamma, a-b]) \\
&= \inf_{\gamma\in\Gamma} \mu([\gamma, a] - [\gamma, b]) \\
&\geq \min[\inf_{\gamma\in\Gamma} \mu([\gamma, a])], \inf_{\gamma\in\Gamma} \mu([\gamma, b])] \\
&= \min[\mu^*(a), \mu^*(b)].
\end{aligned}$$

Again

$$\mu^*(a\alpha b) = \inf_{\gamma\in\Gamma} \mu([\gamma, a\alpha b]) = \inf_{\gamma\in\Gamma} \mu([\gamma, a][\alpha, b]) \geq \inf_{\gamma\in\Gamma} \mu([\gamma, a]) = \mu^*(a).$$

Again

$$\begin{aligned}
\mu^*(a\alpha b) &= \inf_{\gamma\in\Gamma} \mu([\gamma, a\alpha b]) \\
&= \inf_{\gamma\in\Gamma} \mu([\gamma, a][\alpha, b]) \\
&\geq \inf_{\gamma\in\Gamma} \mu([\alpha, b]) \\
&= \mu([\alpha, b]) \\
&\geq \inf_{\gamma\in\Gamma} \mu([\gamma, b]) = \mu^*(b).
\end{aligned}$$

So μ^* is a fuzzy ideal of M. □

Proposition 4.2. *If $\sigma \in FI(M)$ [resp. $FRI(M)$, $FLI(M)$], then $\sigma^{*'} \in FI(R)$ [resp. $FRI(R)$, $FLI(R)$].*

Proof. Let σ be a fuzzy ideal of M. Then $\sigma(0_M) = 1$. Now $\sigma^{*'}([\gamma, 0_M]) = \inf_{m \in M} \sigma(m\gamma 0_M) = \sigma(0_M) = 1$. So $\sigma^{*'}$ is nonempty. Let $\sum\limits_i [\alpha_i, a_i], \sum\limits_j [\beta_j, b_j] \in R$. Then

$$\begin{aligned}
\sigma^{*'}(\sum_i [\alpha_i, a_i] - \sum_j [\beta_j, b_j]) &= \inf_{m \in M} \sigma(\sum_i m\alpha_i a_i - \sum_j m\beta_j b_j) \\
&\geq \inf_{m \in M} [\min[\sigma(\sum_i m\alpha_i a_i), \sigma(\sum_j m\beta_j b_j)]] \\
&= \min[\inf_{m \in M} \sigma(\sum_i m\alpha_i a_i), \inf_{m \in M} \sigma(\sum_j m\beta_j b_j)] \\
&= \min[\sigma^{*'}(\sum_i [\alpha_i, a_i]), \sigma^{*'}(\sum_j [\beta_j, b_j])].
\end{aligned}$$

Again,

$$\begin{aligned}
\sigma^{*'}(\sum_i [\alpha_i, a_i] \sum_j [\beta_j, b_j]) &= \sigma^{*'}(\sum_{i,j} [\alpha_i, a_i\beta_j b_j] \\
&= \inf_{m \in M} \sigma(\sum_{i,j} m\alpha_i a_i \beta_j b_j) \\
&\geq \inf_{m \in M} [\min_i [\sigma(m\alpha_1(\sum_j a_1\beta_j b_j)), \sigma(m\alpha_2(\sum_j a_2\beta_j b_j)), \ldots]] \\
&\geq \inf_{m \in M} [\min_i [\sigma(\sum_j a_1\beta_j b_j), \sigma(\sum_j a_2\beta_j b_j), \ldots]] \\
&= \min[\sigma(\sum_j a_1\beta_j b_j), \sigma(\sum_j a_2\beta_j b_j), \ldots] \\
&\geq \inf_{m \in M} [\sigma(\sum_j m\beta_j b_j)] \\
&= \sigma^{*'}(\sum_j [\beta_j, b_j].
\end{aligned}$$

Similarly we can show that $\sigma^{*'}(\sum\limits_i [\alpha_i, a_i] \sum\limits_j [\beta_j, b_j]) \geq \sigma^{*'}(\sum\limits_i [\alpha_i, a_i])$. So $\sigma^{*'}$ is a fuzzy ideal of R. □

Similarly we can prove the following Proposition.

Proposition 4.3. *If $\delta \in FI(L)$ [resp. $FRI(L)$, $FLI(L)$], then $\delta^+ \in FI(M)$ [resp. $FRI(M)$, $FLI(M)$].*

Proposition 4.4. *If $\eta \in FI(M)$ [resp. $FRI(M)$, $FI(M)$], then $\eta^{+'} \in FI(L)$ [resp. $FRI(L)$, $FLI(L)$].*

Theorem 4.1. *The lattices of all fuzzy ideals (resp. fuzzy left ideals) of M and R are isomorphic via the inclusion preserving bijection $\sigma \to \sigma^{*'}$ where $\sigma \in FI(M)$ [resp. $FLI(M)$] and $\sigma^{*'} \in FI(R)$ [resp. $FLI(R)$].*

Proof. First we shall show that $(\sigma^{*'})^* = \sigma$, where $\sigma \in FI(M)$. Let $a \in M$. Then

$$\begin{aligned}(\sigma^{*'})^*(a) &= \inf_{\gamma\in\Gamma}[\sigma^{*'}([\gamma, a])] \\ &= \inf_{\gamma\in\Gamma}[\inf_{m\in M}[\sigma(m\gamma a)]] \\ &\geq \inf_{\gamma\in\Gamma}[\inf_{m\in M}[\sigma(a)]] = \sigma(a).\end{aligned}$$

So $\sigma \subseteq (\sigma^{*'})^*$. Let $\sum_i [e_i, \delta_i]$ be the left unity of M. Then $\sum_i e_i\delta_i x = x$ for all $x \in M$. Now

$$\begin{aligned}\sigma(a) &= \sigma(\sum_i e_i\delta_i a) \\ &\geq \min_i[\sigma(e_1\delta_1 a), \sigma(e_2\delta_2 a), \ldots])]] \\ &\geq \inf_{\gamma\in\Gamma}[\inf_{m\in M}[\sigma(m\gamma a)]] = (\sigma^{*'})^*(a).\end{aligned}$$

So $(\sigma^{*'})^* \subseteq \sigma$. Hence $\sigma = (\sigma^{*'})^*$. Again, let $\mu \in FI(R)$. Now

$$\begin{aligned}(\mu^*)^{*'}(\sum_k[\alpha_k, a_k]) &= \inf_{m\in M}[\mu^*(\sum_k m\alpha_k a_k)] \\ &= \inf_{m\in M}[\inf_{\gamma\in\Gamma}[\mu(\gamma, \sum_k m\alpha_k a_k)]] \\ &= \inf_{m\in M}[\inf_{\gamma\in\Gamma}[\mu([\gamma, m]\sum_k[\alpha_k, a_k])]] \\ &\geq \mu(\sum_k[\alpha_k, a_k]).\end{aligned}$$

So $\mu \subseteq (\mu^*)^{*'}$. Let $\sum_j [\delta'_j, e'_j]$ be the right unity of M. Then

$$\begin{aligned}\mu(\sum_k[\alpha_k, a_k]) &= \mu(\sum_j[\delta'_j, e'_j]\sum_k[\alpha_k, a_k]) \\ &\geq \min_j[\mu([\delta'_1, e'_1]\sum_k[\alpha_k, a_k]), \mu([\delta'_2, e'_2]\sum_k[\alpha_k, a_k]), \ldots] \\ &\geq \inf_{m\in M}[\inf_{\gamma\in\Gamma}[\mu([\gamma, m]\sum_k[\alpha_k, a_k])]] \\ &= (\mu^*)^{*'}(\sum_k[\alpha_k, a_k]).\end{aligned}$$

So $\mu \supseteq (\mu^*)^{*'}$. Thus $\mu = (\mu^*)^{*'}$. Thus the correspondence $\sigma \to \sigma^{*'}$ is a bijection. Now let $\sigma_1, \sigma_2 \in FI(M)$ be such that $\sigma_1 \subseteq \sigma_2$. Then

$$\begin{aligned}\sigma_1^{*'}(\sum_i[\alpha_i, a_i]) &= \inf_{m\in M}\sigma_1(\sum_i m\alpha_i a_i) \\ &\leq \inf_{m\in M}\sigma_2(\sum_i m\alpha_i a_i) = \sigma_2^{*'}(\sum_i[\alpha_i, a_i])\end{aligned}$$

for all $\sum\limits_i[\alpha_i, a_i] \in R$. So $\sigma_1^{*'} \subseteq \sigma_2^{*'}$. Similarly we can show that if $\mu_1 \subseteq \mu_2$, where $\mu_1, \mu_2 \in FI(R)$, then $\mu_1^* \subseteq \mu_2^*$. So the mapping $\sigma \to \sigma^{*'}$ is a lattice isomorphism. □

Similarly we can prove the following theorem.

Theorem 4.2. *The lattices of all fuzzy ideals (resp. fuzzy right ideals) of M and L are isomorphic via the inclusion preserving bijection $\eta \to \eta^{+'}$, where $\eta \in FI(M)$ [resp. $FRI(M)$] and $\eta^{+'} \in FI(L)$ [resp. $FRI(L)$].*

Theorem 3.2 maybe obtained as a corollary of the above theorems.

Corollary 4.1. *$FRI(M)$ [resp. $FI(M)$, $FLI(M)$] is a complete lattice.*

Proof. The corollary follows from the above theorem and the facts that $FI(M)$, $FRI(M)$ and $FLI(M)$ are complete lattices. □

Theorem 4.3. *A commutative Γ–ring M is a Γ– field if and only if for every fuzzy ideal σ of M, $\sigma(x) = \sigma(y) < \sigma(0_M)$ for all $x, y \in M\backslash\{0_M\}$.*

Proof. Let σ be a fuzzy ideal of M and $\sigma(x) = \sigma(y) < \sigma(0_M)$ for all $x, y \in M\backslash\{0_M\}$. Let $\sum\limits_i[\alpha_i, a_i], \sum\limits_j[\beta_j, b_j] \in R\backslash\{0_R\}$. Then there exist m, m' in M such that $\sum\limits_i m\alpha_i a_i \neq 0_M$ and $\sum\limits_j m'\beta_j b_j \neq 0_M$. Now

$$\sigma^{*'}(\sum_i[\alpha_i, a_i]) = \inf_{m\in M} \sigma(\sum_i m\alpha_i a_i) = \inf_{m\in M} \sigma(\sum_j m\beta_j b_j) = \sigma^{*'}(\sum_j[\beta_j, b_j])$$

(since $\sigma(x) = \sigma(y) < \sigma(0_M)$ for all $x, y \in M\backslash\{0_M\}$). So

$$\sigma^{*'}(\sum_i[\alpha_i, a_i]) = \sigma^{*'}(\sum_j[\beta_j, b_j]) < \sigma^{*'}(0_R)$$

for all $\sum\limits_i[\alpha_i, a_i], \sum\limits_j[\beta_j, b_j] \in R\backslash\{0_R\}$. Let μ be a fuzzy ideal of R and

$$\sum_i[\alpha_i, a_i], \sum_j[\beta_j, b_j] \in R\backslash\{0_R\}.$$

Then

$$\mu(\sum_i[\alpha_i, a_i]) = (\mu^*)^{*'}(\sum_i[\alpha_i, a_i]) = (\mu^*)^{*'}(\sum_j[\beta_j, b_j]) = \mu(\sum_j[\beta_j, b_j]) < \mu(0_R).$$

Also it follows from Lemma 3.4 of [2] that R is commutative. Consequently by Proposition 3.1.10 of [7] it follows that R is a field and hence M is a Γ-field ([2, Theorem 3.5]).

Conversely, suppose that M is a Γ-field and $x, y \in M\backslash\{0_M\}$. Then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $[\gamma_1, x] \neq 0_R$ and $[\gamma_2, y] \neq 0_R$. Let σ be fuzzy ideal of M. Then $\sigma^{*'}$ is a fuzzy ideal of R. Since M is a Γ-field, R is a field. So, by the Proposition 3.1.10 of [7] it follows that

$$\sigma^{*'}(\sum_i[\alpha_i, a_i]) = \sigma^{*'}(\sum_j[\beta_j, b_j]) < \sigma^{*'}(0_R)$$

for all $\sum\limits_i [\alpha_i, a_i], \sum\limits_j [\beta_j, b_j] \in R\backslash\{0_R\}$. Now

$$\sigma(x) = (\sigma^{*'})^*(x) = \inf_{\gamma\in\Gamma} \sigma^{*'}([\gamma, x]) = \inf_{\gamma\in\Gamma} \sigma^{*'}([\gamma, y]) = (\sigma^{*'})^*(y) = \sigma(y) < \sigma(0_M).$$

So $\sigma(x)) = \sigma(y) < \sigma(0_M)$ for all $x, y \in M\backslash\{0_M\}$. □

Definition 4.3. *A commutative Γ-ring M is said to be Noetherian if for every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of ideals of M there exists a positive integer n such that $I_m = I_n$ for all $m \geq n$.*

Theorem 4.4. *A commutative Γ-ring M is Noetherian if every fuzzy ideal of M has finite values.*

Proof. Let M be a commutative Γ-ring and every fuzzy ideal of M have finite values. Let σ be a fuzzy ideal of M. Then since $\sigma^{*'}(\sum\limits_i [\gamma_i, a_i]) = \inf_{m\in M} \sigma(\sum\limits_i m\gamma_i a_i), \sigma^{*'}$ is of finite values whenever σ is of finite values. Let μ be a fuzzy ideal of R. Since for every fuzzy ideal μ of R, $\mu = (\mu^*)^{*'}$, it follows that μ has finite values. So By Theorem 7 of [10] it follows that R is Noetherian and hence M is Noetherian. □

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