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Almost δ-Precontinuous Multifunctions

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Abstract. In 2001, El-Monsef and Nasef have introduced γ-continuous multifunctions and in 2004, Park, Lee and Son have studied δ-precontinuous multifunctions. The purpose of this paper is to generalize some types of continuous multifunctions. In this paper, the notion of almost δ-precontinuous multifunctions is studied. Basic properties, characterizations and relationships of almost δ-precontinuous multifunctions are obtained.

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1. Introduction

Continuity and multifunctions are basic topics in several branches of mathematics such as in general topology, set valued analysis. Several different forms of continuous multifunctions have been introduced and studied over the years. Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. Some of them are semi-continuity [20], α-continuity [16, 25], precontinuity [23], quasi-continuity [22], γ-continuity [2], δ-precontinuity [19] and β-continuity [24].

The aim of this paper is to give a new weaker form of some types of continuity including semi-continuity, α-continuity, precontinuity and δ-precontinuity. In this paper, almost δ-precontinuity is introduced and studied. Moreover, basic properties and preservation theorems of almost δ-precontinuous multifunctions are investigated and relationships between almost δ-precontinuous multifunctions and the other types of continuity are investigated.

In Section 3, the notion of almost δ-precontinuous multifunctions is introduced and characterizations and some relationships of almost δ-precontinuous multifunctions and basic properties of almost δ-precontinuous multifunctions are investigated and obtained. Furthermore, the relationships almost δ-precontinuity and the other types of continuity are investigated. In Section 4, the relationships between almost

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δ-precontinuity and graphs are obtained. In Section 5, the other several properties of almost δ-precontinuity are investigated.

2. Preliminaries

In this paper, spaces (X, τ) and (Y, υ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\text{cl}(A)$ and $\text{int}(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

A subset A of a space X is said to be regular open (respectively regular closed) if $A = \text{int}(\text{cl}(A))$ (respectively $A = \text{cl}(\text{int}(A))$) [28].

The δ-interior [29] of a subset A of X is the union of all regular open sets of X contained in A is denoted by $\delta\text{-int}(A)$. A subset A is called δ-open [29] if $A = \delta\text{-int}(A)$, i. e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [29] if $A = \delta\text{-cl}(A)$, where $\delta\text{-cl}(A) = \{x \in X : A \cap \text{int}(\text{cl}(U)) \neq \emptyset,\ U \in \tau \text{ and } x \in U\}$.

A subset A of a space X is said to be α-open [17] (resp. semi-open [13], preopen [14], b-open [4] or γ-open [11] or sp-open [9], δ-preopen [26], β-open [1] or semi-preopen [3]) if $A \subset \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subset \text{cl}(\text{int}(A))$, $A \subset \text{int}(\text{cl}(A))$, $A \subset \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$, $A \subset \text{int}(\delta\text{-cl}(A))$, $A \subset \text{cl}(\text{int}(\text{cl}(A)))$). The family of all α-open (resp. semi-open, preopen, γ-open, δ-preopen, β-open) sets of X containing a point $x \in X$ is denoted by $\alpha O(X, x)$ (resp. $SO(X, x)$, $PO(X, x)$, $\gamma O(X, x)$, $\delta PO(X, x)$, $\beta O(X, x)$).

The complement of a semi-open (resp. α-open, preopen, β-open, γ-open) set is said to be semi-closed [8] (resp. α-closed [15], preclosed [12]), β-closed [1], γ-closed [11].

The complement of a δ-preopen set is said to be δ-preclosed. The intersection of all δ-preclosed sets of X containing A is called the δ-preclosure [26] of A and is denoted by $\delta\text{-pcl}(A)$. The union of all δ-preopen sets of X contained A is called δ-preinterior of A and is denoted by $\delta\text{-pint}(A)$ [26]. A subset U of X is called a δ-preneighborhood [26] of a point $x \in X$ if there exists a δ-preopen set V such that $x \in V \subset U$. Note that $\delta\text{-pcl}(A) = A \cup \text{cl}(\delta\text{-int}(A))$ and $\delta\text{-pint}(A) = A \cap \text{int}(\delta\text{-cl}(A))$.

The family of all δ-open (resp. δ-preopen, δ-preclosed, regular open, regular closed) sets of X is denoted by $\delta O(X)$ (resp. $\delta PO(X)$, $\delta PC(X)$, $RO(X)$, $RC(X)$).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [5, 7] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \to Y$ is called upper semi continuous (resp. lower semi continuous) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [20].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$ [27].

Definition 2.1. *A multifunction $F : X \to Y$ is said to be:*

(1) *Upper almost continuous* [21, 23, 27] *or upper precontinuous* [23] *(resp. upper quasi-continuous* [22]*, upper α-continuous* [16, 25]*, upper β-continuous* [24]*, upper γ-continuous* [2]*, upper δ-precontinuous* [19]*) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha O(X, x)$, $U \in \beta O(X, x)$, $U \in \gamma O(X, x)$, $U \in \delta PO(X, x)$) such that $F(U) \subset V$.*

(2) *Lower almost continuous* [21, 23, 27] *or lower precontinuous* [23] *(resp. lower quasi-continuous* [22]*, lower α-continuous* [16, 25]*, lower β-continuous* [24]*, lower γ-continuous* [2]*, lower δ-precontinuous* [19]*) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha O(X, x)$, $U \in \beta O(X, x)$, $U \in \gamma O(X, x)$, $U \in \delta PO(X, x)$) such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.*

(3) *Upper (lower) almost continuous or upper (lower) precontinuous (resp. upper (lower) quasi-continuous, upper (lower) α-continuous, upper (lower) β-continuous, upper (lower) γ-continuous, upper (lower) δ-precontinuous) if it has this property at each point of X.*

3. Almost δ-precontinuous multifunctions

In this section, the notion of almost δ-precontinuous multifunctions is introduced and characterizations and some relationships of almost δ-precontinuous multifunctions and basic properties of almost δ-precontinuous multifunctions are investigated and obtained. Furthermore, the relationships almost δ-precontinuity and the other types of continuity are investigated.

Definition 3.1. *A multifunction $F : X \to Y$ is said to be:*

(1) *Lower almost δ-precontinuous at a point $x \in X$ if for each open set V of Y such that $x \in F^-(V)$, there exists a $U \in \delta PO(X, x)$ such that $U \subset F^-(\text{int}(\text{cl}(V)))$,*

(2) *Upper almost δ-precontinuous at a point $x \in X$ if for each open set V of Y such that $x \in F^+(V)$, there exists a $U \in \delta PO(X, x)$ such that $U \subset F^+(\text{int}(\text{cl}(V)))$.*

(3) *Lower (upper) almost δ-precontinuous if F has this property at each point of X.*

The following theorem give some characterizations of an upper almost δ-precontinuous multifunction.

Theorem 3.1. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1) *F is an upper almost δ-precontinuous multifunction,*

(2) *for each $x \in X$ and for each open set V such that $F(x) \subset V$, there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \subset \text{int}(\text{cl}(V))$,*

(3) *for each* $x \in X$ *and for each regular open set* G *of* Y *such that* $F(x) \subset G$*, there exists a* $U \in \delta PO(X, x)$ *such that* $F(U) \subset G$*,*
(4) *for each* $x \in X$ *and for each closed set* K *such that* $x \in F^+(Y \backslash K)$*, there exists a* δ*-preclosed set* H *such that* $x \in X \backslash H$ *and* $F^-(\text{cl}(\text{int}(K))) \subset H$*,*
(5) $F^+(\text{int}(\text{cl}(V))) \in \delta PO(X)$ *for any open set* $V \subset Y$*,*
(6) $F^-(\text{cl}(\text{int}(K))) \in \delta PC(X)$ *for any closed set* $K \subset Y$*,*
(7) $F^+(G) \in \delta PO(X)$ *for any regular open set* G *of* Y*,*
(8) $F^-(K) \in \delta PC(X)$ *for any regular closed set* K *of* Y*,*
(9) *for each point* x *of* X *and each neighbourhood* V *of* $F(x)$*,* $F^+(\text{int}(\text{cl}(V)))$ *is a* δ*-preneighbourhood of* x*,*
(10) *for each point* x *of* X *and each neighbourhood* V *of* $F(x)$*, there exists a* δ*-preneighbourhood* U *of* x *such that* $F(U) \subset \text{int}(\text{cl}(V))$*,*
(11) $\delta\text{-pcl}(F^-(\text{cl}(\text{int}(H)))) \subset F^-(\text{cl}(\text{int}(\text{cl}(H))))$ *for every subset* H *of* Y*,*
(12) $F^+(\text{int}(\text{cl}(\text{int}(N)))) \subset \delta\text{-pint}(F^+(\text{int}(\text{cl}(N))))$ *for every subset* N *of* Y*.*

Proof.

(1)⇔(2): Clear.

(2)⇒(3): Let $x \in X$ and G be a regular open set of Y such that $F(x) \subset G$. By (2), there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \subset \text{int}(\text{cl}(G)) = G$. We obtain $F(U) \subset G$.

(3)⇒(2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $\text{int}(\text{cl}(V)) \in RO(Y)$. By (3), there exists a $U \in \delta PO(X, x)$ such that $F(U) \subset \text{int}(\text{cl}(V))$.

(2)⇒(4): Let $x \in X$ and K be a closed set of Y such that $x \in F^+(Y \backslash K)$. By (2), there exists a $U \in \delta PO(X, x)$ such that $F(U) \subset \text{int}(\text{cl}(Y \backslash K))$. We have

$$\text{int}(\text{cl}(Y \backslash K)) = Y \backslash \text{cl}(\text{int}(K))$$

and

$$U \subset F^+(Y \backslash \text{cl}(\text{int}(K))) = X \backslash F^-(\text{cl}(\text{int}(K))).$$

We obtain $F^-(\text{cl}(\text{int}(K))) \subset X \backslash U$. Take $H = X \backslash U$. Then, $x \in X \backslash H$ and H is a δ-preclosed set.

(4)⇒(2): It can be obtained similarly as (2)⇒(4).

(1)⇒(5): Let V be any open set of Y and $x \in F^+(\text{int}(\text{cl}(V)))$. By (1), there exists $U_x \in \delta PO(X, x)$ such that $U_x \subset F^+(\text{int}(\text{cl}(V)))$. Therefore, we obtain

$$F^+(\text{int}(\text{cl}(V))) = \bigcup_{x \in F^+(\text{int}(\text{cl}(V)))} U_x.$$

Hence, $F^+(\text{int}(\text{cl}(V))) \in \delta PO(X)$.

(5)⇒(1): Let V be any open set of Y and $x \in F^+(V)$. By (5), $F^+(\text{int}(\text{cl}(V))) \in \delta PO(X)$. Take $U = F^+(\text{int}(\text{cl}(V)))$. Then, $F(U) \subset \text{int}(\text{cl}(V))$. Hence, F is upper almost δ-precontinuous.

(5)⇒(6): Let K be any closed set of Y. Then, $Y \backslash K$ is an open set of Y. By (5), $F^+(\text{int}(\text{cl}(Y \backslash K))) \in \delta PO(X)$. Since $\text{int}(\text{cl}(Y \backslash K)) = Y \backslash \text{cl}(\text{int}(K))$, it follows that $F^+(\text{int}(\text{cl}(Y\backslash K))) = F^+(Y\backslash \text{cl}(\text{int}(K))) = X\backslash F^-(\text{cl}(\text{int}(K)))$. We obtain that $F^-(\text{cl}(\text{int}(K)))$ is δ-preclosed in X.

(6)⇒(5): It can be obtained similarly as (5)⇒(6).

(5)⇒(7): Let G be any regular open set of Y. By (5), $F^+(\text{int}(\text{cl}(G))) = F^+(G) \in \delta PO(X)$.

(7)⇒(5): Let V be any open set of Y. Then, $\text{int}(\text{cl}(V)) \in RO(Y)$. By (7), $F^+(\text{int}(\text{cl}(V))) \in \delta PO(X)$.

(6)⇒(8): It can be obtained similarly as (5)⇒(7).

(8)⇒(6): It can be obtained similarly as (7)⇒(5).

(5)⇒(9): Let $x \in X$ and V be a neighbourhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. therefore, we obtain $x \in F^+(G) \subset F^+(V)$. Since $F^+(\text{int}(\text{cl}(G))) \in \delta PO(X)$, $F^+(\text{int}(\text{cl}(V)))$ is a δ-preneighbourhood of x.

(9)⇒(10): Let $x \in X$ and V be a neighbourhood of $F(x)$. By (9), $F^+(\text{int}(\text{cl}(V)))$ is a δ-preneighbourhood of x. Take $U = F^+(\text{int}(\text{cl}(V)))$. Then $F(U) \subset \text{int}(\text{cl}(V))$.

(10)⇒(1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighbourhood of $F(x)$. By (10), there exists a δ-preneighbourhood U of x such that $F(U) \subset \text{int}(\text{cl}(V))$. Therefore, there exists $G \in \delta PO(X)$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset \text{int}(\text{cl}(V))$. We obtain that F is upper almost δ-precontinuous.

(6)⇒(11): For any subset H of Y, $\text{cl}(H)$ is closed in Y. By (6),

$$F^-(\text{cl}(\text{int}(\text{cl}(H))))$$

is δ-preclosed in X. Therefore, we obtain

$$\delta\text{-pcl}(F^-(\text{cl}(\text{int}(H)))) \subset F^-(\text{cl}(\text{int}(\text{cl}(H)))).$$

(11)⇒(6): Let K be any closed set of Y. Then we have

$$\delta\text{-pcl}(F^-(\text{cl}(\text{int}(K)))) \subset F^-(\text{cl}(\text{int}(\text{cl}(K)))) = F^-(\text{cl}(\text{int}(K))).$$

Thus, $F^-(\text{cl}(\text{int}(K)))$ is δ-preclosed in X.

(5)⇒(12): For any subset N of Y, $\text{int}(N)$ is open in Y. By (5),

$$F^+(\text{int}(\text{cl}(\text{int}(N))))$$

is δ-preopen in X. Therefore, we obtain

$$F^+(\text{int}(\text{cl}(\text{int}(N)))) \subset \delta\text{-pint}(F^+(\text{int}(\text{cl}(N)))).$$

(12)⇒(5): Let V be any open set of Y. Then we have

$$F^+(\text{int}(\text{cl}(V))) \subset \delta\text{-pint}(F^+(\text{int}(\text{cl}(V)))).$$

Hence, $F^+(\text{int}(\text{cl}(V)))$ is δ-preopen in X. □

Remark 3.1. For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), the following implications hold:

$$\begin{array}{ccccc} & & & & \text{upper almost} \\ \text{upper semi-continuity} & & & & \delta\text{ -precontinuity} \\ \Downarrow & & & & \Uparrow \\ \text{upper }\alpha\text{-continuity} & \Rightarrow & \text{upper precontinuity} & \Rightarrow & \text{upper }\delta\text{-precontinuity} \\ \Downarrow & & \Downarrow & & \\ \text{upper quasi-continuity} & \Rightarrow & \text{upper }\gamma\text{-continuity} & & \\ & & \Downarrow & & \\ & & \text{upper }\beta\text{-continuity} & & \end{array}$$

Note that none of these implications is reversible. We give an example for the last implication as follows. The other examples can be obtained in [2, 19].

Example 3.1. Let $X = \{a, b, c\}$, $Y = \{-2, -1, 0, 1, 2\}$. Let τ and σ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{-2, -1, 0, 1\}\}$. Define the multifunction $F : X \to Y$ by

$$F(x) = \begin{cases} \{0\}, & \text{if } x = a \\ \{-1, 1\}, & \text{if } x = b \\ \{-2, 2\}, & \text{if } x = c. \end{cases}$$

Then F is upper almost δ-precontinuous but not upper δ -precontinuous, since $\{-2, -1, 0, 1\} \in \sigma$ and $F^+(\{-2, -1, 0, 1\}) = \{a, b\}$ is not δ-preopen in (X, τ).

The following theorem give some characterizations of a lower almost δ-precontinuous multifunction.

Theorem 3.2. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1) *F is a lower almost δ-precontinuous multifunction,*
(2) *for each $x \in X$ and for each open set V such that $F(x) \cap V \neq \emptyset$, there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \cap \operatorname{int}(\operatorname{cl}(V)) \neq \emptyset$,*
(3) *for each $x \in X$ and for each regular open set G of Y such that $F(x) \cap G \neq \emptyset$, there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \cap G \neq \emptyset$,*
(4) *for each $x \in X$ and for each closed set K such that $x \in F^-(Y \backslash K)$, there exists a δ-preclosed set H such that $x \in X \backslash H$ and $F^+(\operatorname{cl}(\operatorname{int}(K))) \subset H$,*
(5) *$F^-(\operatorname{int}(\operatorname{cl}(V))) \in \delta PO(X)$ for any open set $V \subset Y$,*
(6) *$F^+(\operatorname{cl}(\operatorname{int}(K))) \in \delta PC(X)$ for any closed set $K \subset Y$,*
(7) *$F^-(G) \in \delta PO(X)$ for any regular open set G of Y,*
(8) *$F^+(K) \in \delta PC(X)$ for any regular closed set K of Y,*
(9) *$\delta\text{-pcl}(F^+(\operatorname{cl}(\operatorname{int}(H)))) \subset F^+(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(H))))$ for every subset H of Y,*
(10) *$F^-(\operatorname{int}(\operatorname{cl}(\operatorname{int}(N)))) \subset \delta\text{-pint}(F^-(\operatorname{int}(\operatorname{cl}(N))))$ for every subset N of Y.*

Proof. It can be obtained similarly as the previous theorem. □

Lemma 3.1. *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta PO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta PO(X_0)$* [26].

Lemma 3.2. *Let $A \subset X_0 \subset X$. If $X_0 \in \delta O(X)$ and $A \in \delta PO(X_0)$, then $A \in \delta PO(X)$* [26].

Theorem 3.3. *Let $F : X \to Y$ be a multifunction and let U be a δ-open set in X. If F is a lower (upper) almost δ-precontinuous, then the restriction multifunction $F|_U : U \to Y$ is a lower (resp. upper) almost δ-precontinuous.*

Proof. Suppose that V is an open set in Y. Let $x \in U$ and let $x \in (F|_U)^-(V)$. Since F is a lower almost δ-precontinuous multifunction, it follows that there exists a δ-preopen set G such that $x \in G \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$. By Lemma 3.1, we obtain that $x \in G \cap U \in \delta PO(U)$ and $G \cap U \subset (F|_U)^-(\mathrm{int}(\mathrm{cl}(V)))$. Thus, we show that the restriction multifunction $F|_U$ is a lower almost δ-precontinuous.

The proof of the upper almost δ-precontinuity of $F|_U$ is similar to the above. □

Theorem 3.4. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be a δ-open cover of a space X. Then a multifunction $F : X \to Y$ is upper almost δ-precontinuous (resp. lower almost δ-precontinuous) if and only if the restriction $F|_{U_\lambda} : U_\lambda \to Y$ is upper almost δ-precontinuous (resp. lower almost δ-precontinuous) for each $\lambda \in \Lambda$.*

Proof. We prove only the case for F upper almost δ-precontinuous, the proof for F lower almost δ-precontinuous being analogous.

($\Rightarrow$) Let $\lambda \in \Lambda$ and V be any open set of Y. Since F is upper almost δ-precontinuous, $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is δ-preopen in X. By Lemma 3.1,

$$(F|_{U_\lambda})^+(\mathrm{int}(\mathrm{cl}(V))) = F^+(\mathrm{int}(\mathrm{cl}(V))) \cap U_\lambda$$

is δ-preopen in U_λ and hence $F|_{U_\lambda}$ is upper almost δ-precontinuous.

($\Leftarrow$) Let V be any open set of Y. Since $F|_{U_\lambda}$ is upper almost δ-precontinuous for each $\lambda \in \Lambda$, $(F|_{U_\lambda})^+(\mathrm{int}(\mathrm{cl}(V))) = F^+(\mathrm{int}(\mathrm{cl}(V))) \cap U_\lambda$ is δ-preopen in U_λ. By Lemma 3.2, $(F|_{U_\lambda})^+(\mathrm{int}(\mathrm{cl}(V)))$ is δ-preopen in X for each $\lambda \in \Lambda$. We obtain that $F^+(\mathrm{int}(\mathrm{cl}(V))) = \bigcup_{\lambda \in \Lambda} (F|_{U_\lambda})^+(\mathrm{int}(\mathrm{cl}(V)))$ is δ-preopen in X. Hence F is upper almost δ-precontinuous. □

Suppose that (X, τ), (Y, υ) and (Z, ω) are topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 3.5. *Let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. The following statements hold:*

(1) *If F is upper (lower) δ-precontinuous and G is upper (lower) semi-continuous, then $G \circ F : X \to Z$ is an upper (lower) almost δ-precontinuous multifunction.*
(2) *If F is upper (lower) precontinuous and G is upper (lower) semi-continuous, then $G \circ F : X \to Z$ is an upper (lower) almost δ-precontinuous multifunction.*
(3) *If F is upper (lower) α-continuous and G is upper (lower) semi-continuous, then $G \circ F : X \to Z$ is an upper (lower) almost δ-precontinuous multifunction.*
(4) *If F is upper (lower) semi-continuous and G is upper (lower) semi-continuous, then $G \circ F : X \to Z$ is an upper (lower) almost δ-precontinuous multifunction.*

Proof. (1) Let $V \subset Z$ be any regular open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is an upper (lower) semi continuous multifunction, it follows that $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is an upper (lower) δ-precontinuous multifunction, it follows that $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is a δ-preopen set. It shows that $G \circ F$ is an upper (resp. lower) almost δ-precontinuous multifunction.

The other proofs can be obtained similarly. □

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 3.2. *Let (X, τ) be a topological space and let (x_α) be a net in X. It is said that the net (x_α) δ-preconverges to x if for each δ-preopen set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.*

Theorem 3.6. *Let $F : X \to Y$ be a multifunction. If F is a lower (upper) almost δ-precontinuous multifunction, then for each $x \in X$ and for each net (x_α) which δ-preconverges to x in X and for each open set $V \subset Y$ such that $x \in F^-(V)$ (resp. $x \in F^+(V)$), the net (x_α) is eventually in $F^-(\mathrm{int}(\mathrm{cl}(V)))$ (resp. $F^+(\mathrm{int}(\mathrm{cl}(V)))$).*

Proof. Let (x_α) be a net which δ-preconverges to x in X and let V be any open set in Y such that $x \in F^-(V)$. Since F is a lower almost δ-precontinuous multifuction, it follows that there exists a δ-preopen set U in X containing x such that $U \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$. Since (x_α) δ-preconverges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(\mathrm{int}(\mathrm{cl}(V)))$.

The proof of the upper almost δ-precontinuity of F is similar to the above. □

Definition 3.3. *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semi-regularization. In case when $\tau = \tau^*$, the space (X, τ) is called semi-regular* [28].

Theorem 3.7. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a semi-regular topological space (Y, υ). F is a lower almost δ-precontinuous multifunction if and only if F is lower δ-precontinuous.*

Proof. Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y, υ) is a semi-regular space, there exist regular open sets U_i for $i \in I$ such that $V = \bigcup_{i \in I} U_i$. We have $F^-(V) = F^-(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} F^-(U_i)$. By Theorem 3.2, $F^-(U_i) \in \delta PO(X)$ for $i \in I$. We obtain $F^-(V) \in \delta PO(X)$. Hence, by Theorem 3.6 in [19], F is lower δ-precontinuous.

The converse is obvious. □

Corollary 3.1. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then $F : (X, \tau) \to (Y, \upsilon)$ is a lower almost δ-precontinuous multifunction if and only if $F : (X, \tau) \to (Y, \upsilon^*)$ is lower δ-precontinuous.*

Theorem 3.8. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and* *let* $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ *be the projection for each* $\alpha \in J$. *If* F *is an upper (lower) almost* δ*-precontinuous multifunction, then* $P_\alpha \circ F$ *is an upper (resp. lower) almost* δ*-precontinuous multifunction for each* $\alpha \in J$.

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then

$$(P_{\alpha_0} \circ F)^+(\text{int}(\text{cl}(V_{\alpha_0})) = F^+(P^+_{\alpha_0}(\text{int}(\text{cl}(V_{\alpha_0}))) = F^+(\text{int}(\text{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha)$$

and respectively

$$(P_{\alpha_0} \circ F)^-(\text{int}(\text{cl}(V_{\alpha_0})) = F^-(P^-_{\alpha_0}(\text{int}(\text{cl}(V_{\alpha_0}))) = F^-(\text{int}(\text{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha).$$

Since F is an upper (resp. lower) almost δ-precontinuous multifunction and since $\text{int}(\text{cl}(V_{\alpha_0}) \times \prod_{\alpha\neq\alpha_0} X_\alpha$ is a regular open set, it follows that $F^+(\text{int}(\text{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ (respectively, $F^-(\text{int}(\text{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha))$ is δ-preopen in (X, τ). It shows that $P_{\alpha_0} \circ F$ is an upper (lower) almost δ-precontinuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) almost δ-precontinuous multifunction for each $\alpha \in J$. □

Theorem 3.9. *Suppose that for each* $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \upsilon_\alpha)$ *are topological spaces. Let* $F_\alpha : X_\alpha \to Y_\alpha$ *be a multifunction for each* $\alpha \in J$ *and let* $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ *be defined by* $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ *from the product space* $\prod_{\alpha\in J} X_\alpha$ *to the product space* $\prod_{\alpha\in J} Y_\alpha$. *If* F *is an upper (lower) almost* δ*-precontinuous multifunction, then each* F_α *is an upper (resp. lower) almost* δ*-precontinuous multifunction for each* $\alpha \in J$.

Proof. Let $V_\alpha \subseteq Y_\alpha$ be a open set. Then $\text{int}(\text{cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta$ is a regular open set. Since F is an upper (lower) almost δ-precontinuous multifunction, it follows that $F^+(\text{int}(\text{cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F^+_\alpha(\text{int}(\text{cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$ (resp. $F^-(\text{int}(\text{cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F^-_\alpha(\text{int}(\text{cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$) is a δ-preopen set. Consequently, we obtain that $F^+_\alpha(\text{int}(\text{cl}(V_\alpha)))$ (resp. $F^-_\alpha(\text{int}(\text{cl}(V_\alpha))))$ is a δ-preopen set. Thus, we show that F_α is an upper (resp. lower) almost δ-precontinuous multifunction. □

Theorem 3.10. *Suppose that* (X, τ), (Y, υ), (Z, ω) *are topological spaces and* $F_1 : X \to Y$, $F_2 : X \to Z$ *are multifunctions. Let* $F_1 \times F_2 : X \to Y \times Z$ *be a multifunction which is defined by* $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ *for each* $x \in X$. *If* $F_1 \times F_2$ *is an upper (lower) almost* δ*-precontinuous multifunction, then* F_1 *and* F_2 *are upper (resp. lower) almost* δ*-precontinuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is an upper almost δ-precontinuous multifunction, it follows that there exists a δ-preopen set U containing x such that $U \subset (F_1 \times F_2)^+(\text{int}(\text{cl}(K \times H)))$. We obtain that $U \subset F_1^+(\text{int}(\text{cl}(K)))$ and $U \subset F_2^+(\text{int}(\text{cl}(H)))$. Thus, we obtain that F_1 and F_2 are upper almost δ-precontinuous multifunctions.

The proof of the lower almost δ-precontinuity of F_1 and F_2 is similar to the above. □

4. Graphs

In this section, the relationships between almost δ-precontinuity and graphs are investigated.

Lemma 4.1. *Let A be a subset of a space (X, τ). Then $A \in \delta PO(X)$ if and only if $A \cap U \in \delta PO(X)$ for each regular open (δ-open) set U of X* [26].

Lemma 4.2. *For a multifunction $F : X \to Y$, the following hold:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$,

(2) $G_F^-(A \times B) = A \cap F^-(B)$

for any subsets $A \subset X$ and $B \subset Y$ [18].

Theorem 4.1. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$ and X be a semi-regular space. Then the graph multifunction of F is upper almost δ-precontinuous if and only if F is upper almost δ-precontinuous.*

Proof. ($\Rightarrow$). Suppose that $G_F : X \to X \times Y$ is upper almost δ-precontinuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \delta PO(X, x)$ such that $G_F(U) \subset \text{int}(\text{cl}(X \times V)) = X \times \text{int}(\text{cl}(V))$. By the previous lemma, we have $U \subset G_F^+(X \times \text{int}(\text{cl}(V))) = F^+(\text{int}(\text{cl}(V)))$ and $F(U) \subset \text{int}(\text{cl}(V))$. This shows that F is upper almost δ-precontinuous.

($\Leftarrow$): Suppose that $F : X \to Y$ is upper almost δ-precontinuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family of $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, says y_1, y_2, y_3,..., y_n in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) : i = 1, 2, ..., n\}$. Take $U = \bigcap\{U(y_i) : i = 1, 2, ..., n\}$ and $V = \bigcup\{V(y_i) : i = 1, 2, ..., n\}$. Then U and V are open in X and Y, respectively, and since X is semi-regular, there exists a regular open set U_0 such that $\{x\} \times F(x) \subset U_0 \times V \subset U \times V \subset W$. Since F is upper almost δ-precontinuous, there exists $H \in \delta PO(X, x)$ such that $F(H) \subset \text{int}(\text{cl}(V))$. By the previous lemma, we have $U_0 \cap H \subset zU_0 \cap F^+(\text{int}(\text{cl}(V))) = G_F^+(U_0 \times \text{int}(\text{cl}(V))) \subset G_F^+(\text{int}(\text{cl}(U_0 \times V))) \subset G_F^+(\text{int}(\text{cl}(W)))$. Therefore, we obtain $U_0 \cap H \in \delta PO(X, x)$ and $G_F(U_0 \cap H) \subset \text{int}(\text{cl}(W))$. This shows that G_F is upper almost δ-precontinuous. □

Theorem 4.2. *Let X be a semi-regular space. A multifunction $F : X \to Y$ is lower almost δ-precontinuous if and only if $G_F : X \to X \times Y$ is lower almost δ-precontinuous.*

Proof. ($\Rightarrow$) Suppose that F is lower almost δ-precontinuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets U and V of X and Y, respectively. By semi-regularity of X, there exists a regular open set H of X such that $(x, y) \in H \times V \subset U \times V \subset W$. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \delta PO(X, x)$ such that $G \subset F^{-}(\mathrm{int}(\mathrm{cl}(V)))$. By Lemma 4.2, $H \cap G \subset U \cap F^{-}(\mathrm{int}(\mathrm{cl}(V))) = G_F^{-}(U \times \mathrm{int}(\mathrm{cl}(V))) \subset G_F^{-}(\mathrm{int}(\mathrm{cl}(W)))$. Furthermore, $x \in H \cap G \in \delta PO(X)$ and hence G_F is lower almost δ-precontinuous.

($\Leftarrow$) Suppose that G_F is lower almost δ-precontinuous. Let $x \in X$ and V be any open set of Y such that $x \in F^{-}(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost δ-precontinuous, there exists a δ-preopen set U containing x such that $U \subset G_F^{-}(\mathrm{int}(\mathrm{cl}(X \times V)))$. Since $G_F^{-}(\mathrm{int}(\mathrm{cl}(X \times V))) = G_F^{-}(X \times \mathrm{int}(\mathrm{cl}(V)))$, by Lemma 4.2, we have $U \subset F^{-}(\mathrm{int}(\mathrm{cl}(V)))$. This shows that F is lower almost δ-precontinuous. □

Lemma 4.3. *Let A and B be subsets of spaces (X, τ) and (Y, σ), respectively. If $A \in \delta PO(X)$ and $B \in \delta PO(Y)$, then $A \times B \in \delta PO(X \times Y)$* [26].

Theorem 4.3. *If a multifunction $F : X \to Y$ is an upper almost δ-precontinuous multifunction such that $F(x)$ is compact for each $x \in X$ and Y is Hausdorff space, then the multigraph $G(F)$ of F is δ-preclosed in $X \times Y$.*

Proof. $(x, y) \notin G(F)$. That is $y \notin F(x)$. Since Y is Hausdorff, for each $z \in F(x)$, there exist disjoint open sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(z)$. Then $\{U(z) : z \in F(x)\}$ is open cover of $F(x)$ and since $F(x)$ is compact, there exists a finite number of points, say, z_1, z_2, z_3,…, z_n in $F(x)$ such that

$$F(x) \subset \bigcup \{U(z_i) : i = 1, 2, 3, \ldots, n\}.$$

Put

$$U = \bigcup \{U(z_i) : i = 1, 2, 3, \ldots, n\} \text{ and } V = \bigcap \{V(z_i) : i = 1, 2, 3, \ldots, n\}.$$

Then U and V are open in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. Since F is an upper almost δ-precontinuous multifunction, there exists $W \in \delta PO(X, x)$ such that $F(W) \subset \mathrm{int}(\mathrm{cl}(U))$. Since V is open, by Lemma 4.3, it follows that $W \times V \in \delta PO(X \times Y)$ and $(x, y) \in W \times V \subset (X \times Y) \backslash G(F)$. We obtain that

$$(X \times Y) \backslash G(F) = \bigcup_{(x,y) \in (X \times Y) \backslash G(F)} W \times V$$

is δ-preopen in $X \times Y$ and hence $G(F)$ is δ-preclosed in $X \times Y$. □

Definition 4.1. *A subset A of a topological space X is said to be α-paracompact [30] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X. Furthermore, a multifunction $F : (X, \tau) \to (Y, \sigma)$ is called punctually α-paracompact* [25] *if $F(x)$ is α-paracompact for each point $x \in X$.*

Definition 4.2. *Let $F : X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be δ-pre-graph in $X \times Y$ if for each $(x, y) \notin G(F)$, there exist δ-preopen set U and open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.*

Theorem 4.4. *Let $F : (X,\tau) \to (Y,\sigma)$ be an upper almost δ-precontinuous and punctually α-paracompact multifunction into a Hausdorff space (Y,σ). Then the multigraph $G(F)$ of F is a δ-pre-graph in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y,σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members U_{β_1}, U_{β_2},..., U_{β_n} of Φ. Choose y_1, y_2,..., y_n in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, ..., n$ and set $W = W_0 \cap (\bigcap_{i=1}^{n} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\bigcup_{\beta \in I} U_\beta) = \emptyset$, which implies that $W \cap \text{int}(\text{cl}(\bigcup_{\beta \in I} U_\beta)) = \emptyset$. By the upper almost δ-precontinuity of F, there exists a $U \in \delta PO(X, x_0)$ such that $F(U) \subset \text{int}(\text{cl}(\bigcup_{\beta \in I} U_\beta))$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, the graph $G(F)$ is a δ-pre-graph in $X \times Y$. □

5. Some theorems

In this section, the other several properties of almost δ-precontinuity are investigated.

For two multifunctions $F_1 : X \to Y$ and $F_2 : X \to Z$, the product multifunction $F_1 \times F_2 : X \to Y \times Z$ is defined as follows: $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for every $x \in X$.

Theorem 5.1. *If $F_1 \times F_2 : X \to Y \times Z$ is an upper (lower) almost δ-precontinuous multifunction, then $F_1 : X \to Y$ and $F_2 : X \to Z$ are upper (resp. lower) almost δ-precontinuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper almost δ-precontinuous multifunction, it follows that there exists a δ-preopen set U containing x such that $U \subset (F_1 \times F_2)^+(\text{int}(\text{cl}(K \times H)))$. We obtain that $U \subset F_1^+(\text{int}(\text{cl}(K)))$ and $U \subset F_2^+(\text{int}(\text{cl}(H)))$. Hence, we obtain that F_1 and F_2 are upper almost δ-precontinuous multifunctions.

The other proof is similar to the above. □

Definition 5.1. *The δ-prefrontier of a subset A of a space X, denoted by $\delta\text{-pFr}(A)$, is defined by $\delta\text{-pFr}(A) = \delta\text{-pcl}(A) \cap \delta\text{-pcl}(X \backslash A) = \delta\text{-pcl}(A) \backslash \delta\text{-pint}(A)$ [19].*

Theorem 5.2. *The set all points of X at which a multifunction $F : X \to Y$ is not upper almost δ-precontinuous (lower almost δ-precontinuous) is identical with the union of the δ-prefrontier of the upper (lower) inverse images of regular open sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper almost δ-precontinuous. Then there exists a regular open set V of Y containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in \delta PO(X, x)$. Therefore, we have $x \in \delta\text{-pcl}(X \backslash F^+(V)) = X \backslash \delta\text{-pint}(F^+(V))$ and x$\in F^+(V)$. Thus, we obtain $x \in \delta\text{-pFr}(F^+(V))$.

Conversely, suppose that V is a regular open set of Y containing $F(x)$ such that $x \in \delta\text{-pFr}(F^+(V))$. If F is upper almost δ-precontinuous at x, then there exists $U \in \delta PO(X,x)$ such that $U \subset F^+(V)$; hence $x \in \delta\text{-pint}(F^+(V))$. This is a contradiction and hence F is not upper almost δ-precontinuous at x.

The case for lower almost δ-precontinuous is similarly shown. □

In the following $(D,>)$ is a directed set, (F_λ) is a net of multifunction $F_\lambda : X \to Y$ for every $\lambda \in D$ and F is a multifunction from X into Y.

Definition 5.2. *Let $(F_\lambda)_{\lambda\in D}$ be a net of multifunctions from X to Y. A multifunction $F^* : X \to Y$ is defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y$: for each open neighborhood V of y and each $\mu \in D$, there exists $\lambda \in D$ such that $\lambda > \mu$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit of the net $(F_\lambda)_{\lambda\in D}$* [6].

Definition 5.3. *A net $(F_\lambda)_{\lambda\in D}$ is said to be equally upper almost δ-precontinuous at $x_0 \in X$ if for every open set V_λ containing $F_\lambda(x_0)$, there exists a δ-preopen set U containing x_0 such that $F_\lambda(U) \subset \text{int}(\text{cl}(V_\lambda))$ for all $\lambda \in D$.*

Theorem 5.3. *Let $(F_\lambda)_{\lambda\in D}$ be a net of multifunctions from a space X into a compact space Y. If the following are satisfied:*
(1) $\bigcup\{F_\mu(x) : \mu > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$,
(2) $(F_\lambda)_{\lambda\in D}$ is equally upper almost δ-precontinuous on X,
then F^* is upper almost δ-precontinuous on X.

Proof. We have $F^*(x) = \bigcap\{(\bigcup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\}$. Since the net $(\bigcup\{F_\mu(x) : \mu > \lambda\})_{\lambda\in D}$ is a family of closed sets having the finite intersection property and Y is compact, $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper open subset of Y such that $F^*(x_0) \subset V$. Since $F^*(x_0) \cap (Y\backslash V) = \emptyset$, $F^*(x_0) \neq \emptyset$ and $Y\backslash V \neq \emptyset$, $\bigcap\{(\bigcup\{F_\mu(x_0) : \mu > \lambda\}) : \lambda \in D\} \cap (Y\backslash V) = \emptyset$ and hence $\bigcap\{(\bigcup\{F_\mu(x_0) \cap (Y\backslash V) : \mu > \lambda\}) : \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\bigcup\{F_\mu(x_0) \cap (Y\backslash V) : \mu > \lambda\}) : \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that $F_\mu(x_0) \cap (Y\backslash V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda\in D}$ is equally upper almost δ-precontinuous on X, there exists a δ-preopen set U containing x_0 such that $F_\mu(U) \subset \text{int}(\text{cl}(V))$ for each $\mu > \lambda$, i.e., $F_\mu(x) \cap (Y\backslash \text{int}(\text{cl}(V))) = \emptyset$ for each $x \in U$. Then we have $\bigcup\{F_\mu(x) \cap (Y\backslash \text{int}(\text{cl}(V))) : \mu > \lambda\} = \emptyset$ and hence $\bigcap\{(\bigcup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\} \cap (Y\backslash \text{int}(\text{cl}(V))) = \emptyset$. This implies that $F^*(U) \subset \text{int}(\text{cl}(V))$. If $V = Y$, then it is clear that for each δ-preopen set U containing x_0 we have $F^*(U) \subset \text{int}(\text{cl}(V))$. Hence F^* is upper almost δ-precontinuous at x_0. Since x_0 is arbitrary, the proof completes. □

Recall that a multifunction $F : X \to Y$ is said to be punctually connected if, for each $x \in X$, $F(x)$ is connected.

Definition 5.4. *A space X is called δ-preconnected provided that X is not the union of two disjoint nonempty δ-preopen sets* [10].

Theorem 5.4. *Let F be a multifunction from a δ-preconnected topological space X onto a topological space Y such that F is punctually connected. If F is an upper almost δ-precontinuous multifunction, then Y is a connected space.*

Proof. Let $F : X \to Y$ be an upper almost δ-precontinuous multifunction from a δ-preconnected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is an upper almost δ-precontinuous multifunction, $F^+(H)$ and $F^+(K)$ are δ-preopen subsets of X. In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the δ-preconnectedness of X. Hence, it is obtained that Y is a connected space. □

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if, for each $x \in X$, $F(x)$ is closed.

Definition 5.5. *A multifunction $F : X \to Y$ is said to be lower (resp. upper) R-multifunction if $F^-(V)$ (resp. $F^+(V)$) is a regular open set in X for any regular open set $V \subset Y$.*

Theorem 5.5. *Let F be an upper almost δ-precontinuous punctually closed multifunction and G be a punctually closed upper R-multifunction from a topological space X to a normal topological space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is δ-preclosed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a normal space, it follows that there exists disjoint open sets U and V containing $F(x)$ and $G(x)$ respectively. Since F is upper almost δ-precontinuous and G is an upper R-multifunction, then the sets $F^+(\mathrm{int}(\mathrm{cl}(U)))$ and $G^+(\mathrm{int}(\mathrm{cl}(V)))$ are δ-preopen and regular open, respectively such that contain x. Let $H = F^+(\mathrm{int}(\mathrm{cl}(U))) \cap G^+(\mathrm{int}(\mathrm{cl}(V)))$. By Lemma 4.1, H is a δ-preopen set containing x and $H \cap K = \emptyset$. Hence, K is δ-preclosed in X. □

Theorem 5.6. *If Y is normal space and $F_i : X_i \to Y$ is an upper almost δ-precontinuous multifunction such that F_i is punctually closed for $i = 1, 2$, then a set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is δ-preclosed set in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1, x_2) \in (X_1 \times X_2)\backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is normal and F_i is punctually closed for $i = 1, 2$, there exist disjoint open sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper almost δ-precontinuous, $F_i^+(\mathrm{int}(\mathrm{cl}(V_i)))$ is δ-preopen for $i = 1, 2$. Put $U = F_1^+(\mathrm{int}(\mathrm{cl}(V_1))) \times F_2^+(\mathrm{int}(\mathrm{cl}(V_2)))$, then U is δ-preopen and $(x_1, x_2) \in U \subset (X_1 \times X_2)\backslash A$. This shows that $(X_1 \times X_2)\backslash A$ is δ-preopen and hence A is δ-preclosed in $X_1 \times X_2$. □

Definition 5.6. *A space X is said to be δ-pre-T_2 (δ-pre-Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint δ-preopen sets U and V in X such that $x \in U$ and $y \in V$* [10].

Theorem 5.7. *Let $F : X \to Y$ be an upper almost δ-precontinuous multifunction and punctually closed from a topological space X to a normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a δ-pre-Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a normal space, it follows that there exists disjoints open sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(\text{int}(\text{cl}(U)))$ and $F^+(\text{int}(\text{cl}(V)))$ are disjoint δ-preopen sets containing x and y respectively. Thus, it is obtained that X is δ-pre-Hausdorff. □

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