BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **28***(1) (2005), 47–53*

Some Results on Value Distribution of Meromorphic Functions

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Abstract. In this paper, we study the value distribution of meromorphic functions on plane domains.

2000 Mathematics Subject Classification: 30D35, 30D45

Key words and phrases: Meromorphic function, normal family, derivatives.

1. Introduction

Let f be a nonconstant meromorphic function. In this paper, we use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), N(r,\frac{1}{f}), \cdots$$

(see Schiff [5], Yang [7]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to \infty$, possibly outside of a set with finite measure.

Let D be a domain in $\mathbb{C}$, and let $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sence of Montel, if for every sequence f_n there exists a subsequence f_{n_j}, such that f_{n_j} spherically converges, locally uniformly in D, to a meromorphic function or ∞ (see [5]).

A meromorphic function f on $\mathbb{C}$ is called a normal function if there exist a positive number M such that

$$f^\sharp(z) \le M.$$

Here, as usual, $f^\sharp(z) = \dfrac{|f'(z)|}{1+|f(z)|^2}$ denotes the spherical derivative of $f(z)$.

In the present paper, we study the value distribution of meromorphic function on plane domains. As a first result, we have

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ. Let a be a finite non-zero complex number and k be a positive integer. If for every function $f \in \mathcal{F}$, f has no zeros, and $ff^{(k)} \ne a$, then $\mathcal{F}$ is normal on Δ.*

Received: September 27, 2003; *Revised:* December 12, 2003.

Remark 1.1. The following example show that $a \neq 0$ is necessary in Theorem 1.1.

Example 1.1. Let $\mathcal{F}$=$\{f_n\}$, where $f_n(z) = e^{nz}$. Then

$$f_n(z)f_n^{(k)}(z) = n^k e^{2nz}.$$

Obviously,

$$f_n(z)f_n^{(k)}(z) \neq 0, \quad f_n(z) \neq 0.$$

But $\mathcal{F}$ is not normal on the unit disc Δ.

Moreover, we have

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ. Let a be a finite non-zero complex number and k be a positive integer. If for every function $f \in \mathcal{F}$, f has no zeros, and $|f^{(k)}| \leq M$ whenever $ff^{(k)} = a$, $M > 0$, then $\mathcal{F}$ is normal on Δ.*

In order to prove Theorem 1.1 and Theorem 1.2, we will first prove

Theorem 1.3. *Let f be a meromorphic function all of whose zeros have multiplicity at least k and k be a positive integer. If $N(r, \frac{1}{f}) = S(r, f)$, then $ff^{(k)}$ takes on every nonzero finite value $a \in \mathbb{C}$ infinitely often.*

Corresponding to Theorem 1.1, we also get the following results on normal function.

Theorem 1.4. *Let f be a meromorphic function on $\mathbb{C}$, let a be a finite non-zero complex number and k be a positive integer. If f has no zeros, and $ff^{(k)} \neq a$, then f is a normal function on $\mathbb{C}$.*

In the second part of this paper, we shall prove the following results.

Theorem 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and $h(z)$ be a continuous function in D such that $h(z) \neq 0$ for $z \in D$. If for each $f \in \mathcal{F}$, $f \neq 0$ and $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

As an immediate consequence, we have the

Corollary 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and $h(z)$ be a non-vanishing analytic function in D such that $h(z) \neq 0$ for $z \in$$D$. If for each $f \in \mathcal{F}$, $f \neq 0$ and $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

However, requiring that $h(z)$ have no multiple poles for $z \in D$, we have

Theorem 1.6. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $h(z)$ be a meromorphic function in D such that $h(z) \not\equiv 0$ and $h(z)$ have no multiple poles for $z \in D$. If for each $f \in \mathcal{F}$, $f \neq 0$, and $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

Remark 1.2. The hypothesis that $h(z)$ have no multiple poles for $z \in D$ in Theorem 1.6 is necessary as is shown by the following example.

Example 1.2. Let $D = \{z : |z| < 1\}$. k is a positive integer, and $\mathcal{F}$=$\{f_n\}$, where

$$f_n(z) = \frac{1}{nz^k}, \quad h(z) = \frac{1}{z^{k+1}}, \quad n = 1, 2, 3 \cdots .$$

Clearly, $\mathcal{F}$ fails to be normal at $z = 0$. Obviously, $f \neq 0$ and $f_n'(z) \neq h(z)$, but the poles of $h(z)$ are of multiplicity ≥ 2.

2. Some lemmas

For the proof of our results, we require the following lemmas.

Lemma 2.1. [6] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, for every $f \in \mathcal{F}$, f has no zeros. then if $\mathcal{F}$ is not normal, there exist, for each $\alpha > 0$,*

(a) *a number r, $0 < r < 1$,*
(b) *points z_n, $|z_n| < r$,*
(c) *functions $f_n \in \mathcal{F}$,*
(d) *positive numbers $\varrho_n \to 0$,*

such that

$$\frac{f_n(z_n + \varrho_n \xi)}{\varrho_n^\alpha} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbf{C}$ such that $g^\sharp(\xi) \leq g^\sharp(0) = 1$, g has no zeros.

Lemma 2.2. [3] *Let $f(z)$ be a meromorphic function, let a be a non-zero complex number and let k be a positive integer. If $f(z) \neq 0$, $f^{(k)}(z) \neq a$, then $f(z)$ is a constant.*

Lemma 2.3. [1] (cf. [2]) *Let M be the set of all triples (ϕ, U, ω), where U is a bounded open subset of $\mathbb{C}$, $\phi : \overline{U} \to Z$ such that*

(i) *if U is a piecewise-smoothly Jordan domain and ϕ is holomorphic on $\overline{U}$, then $d(\phi, U, \omega)$ is the winding number of $\phi(\partial U)$ about ω(and hence, by the argument principle, the number of times ϕ takes on the value ω in U;*
(ii) *if $\phi : \overline{U} \to \mathbb{C}$ is a continuous function such that $|\varphi(\xi) - \phi(\xi)| < dist(\omega, \phi(\partial U))$ for each $\xi \in \overline{U}$, then $d(\phi, U, \omega)$; and*
(iii) *if $d(\phi, U, \omega) \neq 0$, then $\overline{U} \bigcap \phi^{-1}(\omega) \neq \emptyset$.*

Lemma 2.4. [4] *Let f be a meromorphic function and $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}\left(r, f\right) + S(r, f).$$

3. Proof of Theorem 1.3

We consider function $\varphi = f^{(k)} f$. Clearly, all poles of φ have multiplicity at least $k + 2$, so

$$\bar{N}\left(r, \varphi\right) \leq \frac{N\left(r, \varphi\right)}{k + 2}.$$

From Lemma 2 we can deduce that

$$\begin{aligned}\bar{N}\left(r,\frac{1}{\varphi}\right) &\leq N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}}\right)\\ &\leq 2N\left(r,\frac{1}{f}\right)+k\bar{N}\left(r,f\right)+S(r,f)\\ &= k\bar{N}(r,\varphi)+S(r,f).\end{aligned}$$

The second fundamental theorem now implies that

$$\begin{aligned}T(r,\varphi) &\leq \bar{N}(r,\varphi)+\bar{N}\left(r,\frac{1}{\varphi}\right)+\bar{N}\left(r,\frac{1}{\varphi-a}\right)+S(r,f)\\ &\leq (k+1)\bar{N}(r,\varphi)+\bar{N}\left(r,\frac{1}{\varphi-a}\right)+S(r,f)\\ &\leq \frac{k+1}{k+2}N(r,\varphi)+\bar{N}\left(r,\frac{1}{\varphi-a}\right)+S(r,f)\\ &\leq \frac{k+1}{k+2}T(r,\varphi)+\bar{N}\left(r,\frac{1}{\varphi-a}\right)+S(r,f)\end{aligned}$$

so that

$$\bar{N}\left(r,\frac{1}{\varphi-a}\right)\geq\frac{T\left(r,\varphi\right)}{k+2}-S(r,f).$$

Hence $\varphi-a$ have infinitely many zeros.

4. Proof of Theorem 1.1

Suppose not. Then by Lemma 2.1, there exists $f_n\in\mathcal{F}$, $z_n\in\Delta$ and $\varrho_n\to 0^+$ such that

$$\varrho_n^{-\frac{k}{2}}f_n(z_n+\varrho_n\xi)=g_n(\xi)\to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function and g has no zeros. Then

$$f_n^{(k)}(z_n+\varrho_n\xi)f_n(z_n+\varrho_n\xi)=g_n^{(k)}(\xi)g_n(\xi)\to g^{(k)}(\xi)g(\xi).$$

Since

$$f_n^{(k)}(z_n+\varrho_n\xi)f_n(z_n+\varrho_n\xi)\neq a,$$

by Hurwitz's theorem we can derive that

(i) $g^{(k)}g\equiv a$,
(ii) $g^{(k)}g\neq a$.

If $g^{(k)}g\equiv a$, since $g\neq 0$, so g is an entire function and hence of exponential type. Hence $g(\xi)=Ae^{c\xi}$, where $A\neq 0, c\neq 0$. But then $g(\xi)g^{(k)}(\xi)=c^kA^2e^{2c\xi}$, which contradicts $g^{(k)}g\equiv a$, Thus (i) is impossible. So $g^{(k)}g\neq a$, but it reduces a contradiction from Theorem 1.3. The contradiction establishes the Theorem.

5. Proof of Theorem 1.2

Suppose not. Then by Lemma 2.1, there exist $f_n \in \mathcal{F}$, $z_n \in \Delta$ and $\varrho_n \to 0^+$ such that
$$\varrho_n^{-\frac{k}{2}} f_n(z_n + \varrho_n \xi) = g_n(\xi) \to g(\xi)$$
locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function and g has no zeros. Then
$$f_n^{(k)}(z_n + \varrho_n \xi) f_n(z_n + \varrho_n \xi) = g_n^{(k)}(\xi) g_n(\xi) \to g^{(k)}(\xi) g(\xi).$$
From Theorem 1.3, there exists ξ_0 such that $g^{(k)}(\xi_0)g(\xi_0) = a$. Clearly $g^{(k)}g \not\equiv a$, then by Hurwitz's theorem, there exist ξ_n, $\xi_n \to \xi_0$, such that (for n large enough)
$$g_n^{(k)}(\xi_n) g_n(\xi_n) = f_n^{(k)}(z_n + \varrho_n \xi_n) f_n(z_n + \varrho_n \xi_n) = a.$$
By assumption, we have
$$|g_n^{(k)}(\xi_n)| = \varrho_n^{\frac{k}{2}} |f_n^{(k)}(z_n + \varrho_n \xi_n)| \le \varrho_n^{\frac{k}{2}} M.$$
Hence
$$|g^{(k)}(\xi_0)| = \lim_{n\to\infty} |g_n^{(k)}(\xi_n)| \le 0,$$
Thus $g^{(k)}(\xi_0) = 0$, which contradicts $g^{(k)}(\xi_0)g(\xi_0) = a \neq 0$. This proved Theorem 1.2.

6. Proof of Theorem 1.4

Suppose f is not a normal function. Then there exist $z_n \to \infty$ such that
$$\lim_{n\to\infty} f^{\sharp}(z_n) = \infty.$$
Write $f_n(z) = f(z + z_n)$ and set $\mathcal{F} = \{f_n\}$. Then by Marty's criterion, $\mathcal{F}$ is not normal on the unit disc. On the other hand, since f_n has no zeros, and $f_n f_n^{(k)} \neq a$, Theorem 1.1 implies that $\mathcal{F}$ is normal. The contradiction proves the theorem.

7. Proof of Theorem 1.5

Since normality is a local property, we may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is normal on Δ. Then by Lemma 2.1, there exist $f_n \in \mathcal{F}$, $z_n \in D$, and $\varrho_n \to 0^+$ such that
$$g_n(\xi) = \frac{f_n(z_n + \varrho_n \xi)}{\varrho_n} \to g(\xi)$$
locally uniformly with respect to the spherical metric , where g is a nonconstant meromorphic function, g has no zeros. Taking a subsequence and renumbering, we may assume that $z_n \to z_0 \in \Delta$.

We claim $g'(\xi) \neq h(z_0)$.

Clearly, $g'(\xi) \not\equiv h(z_0)$, since otherwise g would be linear , which contradicts that $g \neq 0$. Suppose $g'(\xi_0) = h(z_0)$. Then $\phi = g' - h(z_0)$ is a nonconstant analytic function on a neighborhood V of ξ_0, which vanishes at ξ_0. Let $\Delta_\varepsilon = \{\omega : |\omega| < \varepsilon\}$. For $\varepsilon > 0$ sufficiently small, the component U of $\phi^{-1}(\Delta_\varepsilon)$ containing ξ_0 is relatively compact in V and satisfies $\phi(\partial U) = \{\omega : |\omega| = \varepsilon\}$ and $d(\phi, U, 0) > 0$, where d is the local degree. Set $\phi_n(\xi) = g'_n(\xi) - h(z_n + \varrho_n\xi)$; then $\phi_n \to \phi$ locally uniformly on V. Thus, for n large enough, we have $|\phi_n(\xi) - \phi(\xi)| < \varepsilon$ on $\overline{U}$. By (ii) of Lemma 2.3, $d(\phi_n, U, 0) = d(\phi, U, 0) > 0$, so that by (iii) of the same result, there exists $\xi_1 \in \overline{U}$ such that $\phi_n(\xi_1) = 0$. But this contradicts $f'_n(z) \neq h(z)$ on Δ. The claim is proved.

Since $g'(\xi) \neq h(z_0)$, it follows from Lemma 2.2 that g must be a constant, which is a contradiction.

8. Proof of Theorem 1.6

We may assume that $D = \Delta$, the unit disc. Normality is a local property, hence it is enough to show that $\mathcal{F}$ is normal at each $z_0 \in D$. We distinguish two cases.

Case (1): $h(z_0) \neq 0, \infty$. Then by Corollary 1.1, we know that $\mathcal{F}$ is normal at z_0.

Case (2): $h(z_0) = 0$ or $h(z_0) = \infty$. Then there exists δ, $0 < \delta < 1$, such that $U_{z_0}(\delta) = \{z : |z - z_0| < \delta\} \subset D$. Clearly, $h(z) \neq 0, \infty$ for all $z \in U_{z_0}(\delta) \setminus \{z_0\}$. By case (1), $\mathcal{F}$ is normal there.

Then for each sequence of functions $f_n \in \mathcal{F}$, f_n has a subsequence (without loss of generality, we may take f_n itself), f_n converges to ϕ uniformly on any compact subsects in $U_{z_0}(\delta) \setminus \{z_0\}$ (where ϕ is meromorphic function or ∞). Since $f_n \neq 0$, by Hurwitz's Theorem, we derive that $\phi(z) \equiv 0$ or $\phi(z) \neq 0$.

If $\phi(z) \not\equiv 0$, then $\phi(z) \neq 0$. Otherwise $\phi(z) \equiv 0$, which is a contradiction. Thus there exists a positive number M, such that

$$|f_n(z)| \geq M, \quad |z - z_0| = \frac{\delta}{2} \quad (n \text{ large enough}).$$

Since $f_n(z) \neq 0$ on $U_{z_0}(\delta) \setminus \{z_0\}$ then

$$|f_n(z)| \geq M, |z - z_0| < \frac{\delta}{2}.$$

Thus $\mathcal{F}$ is normal at $z = z_0$.

If $\phi(z) \equiv 0$, then $f_n(z)$ converges uniformly to 0 in $K = \{z : \frac{1}{4}\delta \leq |z - z_0| \leq \frac{3}{4}\delta\}$, so does f'_n . Since $h(z) \not\equiv 0$ in D, we can deduce that there exists $M > 0$ such that $|h(z)| \geq M$ in K. Thus $\frac{f'_n}{h}$ converges uniformly to 0 in K, so is $(\frac{f'_n}{h})'$. Since $f'_n(z) \neq h(z)$ and $h(z)$ has no multiple poles, we have $\frac{f'_n}{h} - 1 \neq 0$(in D). By $n(w = a, m_0, r)$ we denote the number of zeros of $w - a$ counting multiplicity in the disk $U_r(m_0)$. Thus

$$n\left(\frac{1}{\frac{f'_n}{h} - 1}, z_0, \frac{\delta}{2}\right) = 0$$

and

$$\left|n\left(\frac{f'_n}{h} - 1, z_0, \frac{\delta}{2}\right) - n\left(\frac{1}{\frac{f'_n}{h} - 1}, z_0, \frac{\delta}{2}\right)\right|$$
$$= \left|\frac{1}{2\pi i}\int_{|z-z_0|=\frac{\delta}{2}} \frac{(\frac{f'_n}{h} - 1)'}{\frac{f'_n}{h} - 1} dz\right| \to 0 \quad (n \to \infty)$$

which implies that

$$n\left(\frac{f_n'}{h}-1, z_0, \frac{\delta}{2}\right)=0 \quad (n \text{ large enough}).$$

Thus f_n' is holomorphic in $|z-z_0|<\dfrac{\delta}{2}$ for n sufficiently large, and so is f_n. By the maximum principle, it follows that $f_n \to 0$ locally uniformly on compact subsets of $U_{z_0}(\frac{\delta}{2})=\left\{z : |z-z_0|<\dfrac{\delta}{2}\right\}$. Hence $\mathcal{F}$ is normal at z_0 and the proof is completed.

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