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On Differential Rational Invariants of Finite Subgroups of Affine Group

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Abstract. In the case of differential field an algebraic analogue of the field of all invariant differential rational functions of surfaces relative to gauge transformations and different subgroups of affine motion group is introduced. In the case of finite subgroup of affine group it is proved that its field of invariant rational functions generates the above field of invariant differential rational functions as a some differential field.

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1. Introduction

The following is one of the main problems in differential geometry. Let n, m be any natural numbers and H be a subgroup of affine group $GL(n+m, R) \propto R^{n+m}$, $G = \mathrm{Diff}(B)$ be the group of diffeomorphisms of the open unit ball $B \subset R^n$, and $u : B \to R^{n+m}$ be a surface, where u is considered to be infinitely smooth.

A function $f^{\partial}(u(t))$ of $u(t) = (u_1(t), \dots, u_{n+m}(t))$ and its finite number of derivatives relative to $\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_n}$ is said to be invariant (more exactly, (G, H)-invariant) if the equality

$$f^{\partial}(u(t)) = f^{\delta}(u(t(s))h + h_0)$$

is valid for any $t(s) \in G$, $(h, h_0) \in H$ and $s \in B$, where $\delta_i = \frac{\partial}{\partial s_i}$.

Description of all such invariant functions is one of the main problems in equivalence problem of surfaces relative to motion group H of the vector space R^{n+m} and gauge transformations G. As such a problem it was an object of boundless number of investigations. Differential geometry of any motion group H had to deal with this problem.

The above differential operators $\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_n}$ and $\frac{\partial}{\partial s_1}, \dots \frac{\partial}{\partial s_n}$ are related in the following way

$$\frac{\partial}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial s_j(t)}{\partial t_i} \frac{\partial}{\partial s_j},$$

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where $s(t) = (s_1(t), \ldots, s_n(t))$ is the inverse of $t(s) \in G$, i.e., $\delta = g^{-1}\partial$, where g is the matrix with the elements $g_{ij} = \frac{\partial s_j(t)}{\partial t_i}$ at $i, j = \overline{1, n}$. $\partial(\delta)$ is the column vector with the "coordinates" $\frac{\partial}{\partial t_1}, \ldots \frac{\partial}{\partial t_n}$ (respectively $\frac{\partial}{\partial s_1}, \ldots \frac{\partial}{\partial s_n}$).

Let t run B and $F = C^\infty(B)$ be the differential ring of infinitely smooth functions relative to differential operators $\frac{\partial}{\partial t_1}, \ldots \frac{\partial}{\partial t_n}$. Every infinitely smooth surface $u : B \to R^{n+m}$ can be considered as an element of differential module $(F^{n+m}; \partial_1, \ldots, \partial_n)$ where $\partial_i = \frac{\partial}{\partial t_i}$ acts on elements of F^{n+m} coordinate-wisely. If elements of this module are considered as row vectors the above transformations look like $u = (u_1, \ldots, u_{n+m}) \mapsto uh + h_0$, $\partial \mapsto g^{-1}\partial$ where $g_{ij} = \frac{\partial s_j(t)}{\partial t_i}$ at $i, j = \overline{1, n}$, $s(t) \in G$, $(h, h_0) \in H$.

Here is a natural algebraization of the above situation by the use of differential algebra. The used in this paper notations and results of differential algebra can be found, for example, in [2].

Let $(F; \partial_1, \ldots, \partial_n)$ be a differential field, $K = \{a \in F : \partial_i a = 0 \text{ at } i = \overline{1, n}\}$ be its constant field, H be any subgroup of $GL(n+m, K) \propto K^{n+m}$, ∂ stand for the column-vector of commuting system of differential operators $\partial_1, \ldots, \partial_n$ and

$$GL^\partial(n, F) = \{g \in GL(n, F) : \partial_i g_{jk} = \partial_j g_{ik} \text{ for } i, j, k = \overline{1, n}\}.$$

It can be checked that components of $\delta = g^{-1}\partial$ for every $g \in GL^\partial(n, F)$ also commute with each other. Let $x_1, \ldots, x_{n+m}$ be differential algebraic indeterminates over K, $x = (x_1, \ldots, x_{n+m})$ and $K^\partial\langle x\rangle$ be the ring of all ∂-rational functions in x over K, where abbreviation "∂-" means "$(\partial_1, \ldots, \partial_n)$-differential."

Definition 1.1. *An element $f^\partial\langle x\rangle \in K^\partial\langle x\rangle$ is said to be $(GL^\partial(n, F), H)$-invariant ($GL^\partial(n, F)$-invariant; H-invariant) if the equality*

$$f^{g^{-1}\partial}\langle xh + h_0\rangle = f^\partial\langle x\rangle \ \left(\textit{respectively } f^{g^{-1}\partial}\langle x\rangle = f^\partial\langle x\rangle; f^\partial\langle xh + h_0\rangle = f^\partial\langle x\rangle\right)$$

is valid for any $g \in GL^\partial(n, F)$, $(h, h_0) \in H$.

Let $K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}$ ($K^\partial\langle x\rangle^{GL^\partial(n,F)}$, $K^\partial\langle x\rangle^H$) stand for all $(GL^\partial(n, F), H)$-invariant (respectively $GL^\partial(n, F)$-invariant, H-invariant) ∂-rational functions in x over K.

Description of the field $K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}$ can be considered as an algebraic analogue of the above geometric problem. The case of paths (curves) for different groups H was an object of many investigations. They are summed up in [1, 4]. A description of above field in the hypersurface affine case is given in [3]. In this paper we investigate the case when H is finite. We will show that in this case the field $K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}$ can be generated by ordinary H invariant rational functions as a some differential field.

Usually $(GL^\partial(n, F), H)$-equivalence can be reduced to H-equivalence [3]. Therefore first we investigate H-invariant differential rational functions then come back to $(GL^\partial(n, F), H)$-invariant case.

2. H-invariant differential rational functions

Let $K(x)$ ($K(x)^H$) stand for the field of all (respect. H-invariant) rational functions in x over K. It is clear that $K(x)^H \subset K^\partial\langle x\rangle^H$. So far as $(K^\partial\langle x\rangle^H; \partial_1, \ldots, \partial_n)$ is a ∂-field we are interested in if $K(x)^H$ generates $K^\partial\langle x\rangle^H$ as a ∂-field over K. As to the field $K(x)^H$ it is an important object in invariant theory and it is studied relatively well for different groups H.

It is well known that extension $K^\partial\langle x\rangle^H \subset K^\partial\langle x\rangle$ is a differential algebraic extension [1] i.e.

$$\operatorname{Diff.tr.deg} K^\partial\langle x\rangle^H/K = n+m.$$

Therefore if $\operatorname{tr.deg} K(x)^H/K < n+m$, then $K(x)^H$ is not able to generate $K^\partial\langle x\rangle^H$ as a ∂-field. In other words the condition $\operatorname{tr.deg} K(x)^H/K = n+m$ is a necessary condition for $K(x)^H$ to generate $K^\partial\langle x\rangle^H$ as a ∂-field. One of the main aims of this paper is to prove that this condition is sufficient as well. Moreover, we will show that it happens if and only if H is a finite group. Let us prove the following key lemma which in its turn relies on Artins Theorem [5].

Lemma 2.1. *Let $(D;\partial_1,\ldots,\partial_n)$ be any differential field, H be any finite group of automorphisms of $(D;\partial_1,\ldots,\partial_n)$. If $L \subset D$ such a H-invariant subfield generating D as a ∂-field then L^H also generates D^H as such a differential field.*

Proof. We will prove this Lemma by induction on n and therefore let us consider first the case $n=1$. Due to Artins Theorem, the field L is a finite Galois extension of L^H and $q = |L : L^H| \le |H|$. Let α be a primitive element of L over L^H i.e.

$$L = \operatorname{Lin}_{L^H}(1,\alpha,\ldots,\alpha^{q-1}) = L^H + \alpha L^H + \cdots + \alpha^{q-1}L^H.$$

Consider the following tower of sets

$$L = D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots,$$

where $D_k = \operatorname{Lin}_{L_K}(1,\alpha,\ldots,\alpha^{q-1})$ and $L_k = (L_H, \partial_1 L^H, \ldots, \partial_1^k L^H)$ – the subfield of D generated by elements of the form $a_0, \partial_1 a_1, \ldots, \partial_1^k a_k$, where $a_i \in L^H$, $i = \overline{0,k}$. The above tower of sets is a tower of fields because α is algebraic over L^H. It is not difficult to see that $\partial_1 D_k \subset D_{k+1}$ for any whole number k. Indeed, the inclusion $\partial_1 L_k \subset L_{k+1}$ is evident and if $p[t]$ stands for minimal polynomial of α over L^H then

$$\frac{\partial p[\alpha]}{\partial t}\partial_1\alpha + p^{\partial_1}[\alpha] = 0 \text{ and } \frac{\partial p[\alpha]}{\partial t} \neq 0,$$

where $p^{\partial_1}[t]$ stands for the polynomial obtained from $p[t]$ by replacing all its coefficients by their ∂_1-derivatives. Therefore $p^{\partial_1}[\alpha] \in D_1$ and the above equality implies $\partial_1\alpha \in D_1$. Hence $\partial_1 D_k \subset D_{k+1}$ and $\partial_1^{k+1} L \subset D_{k+1}$. Therefore due to the condition of the Lemma

$$D = \cup_{k=0}^{\infty} D_k = \cup_{k=0}^{\infty} \operatorname{Lin}_{L_k}(1,\alpha,\ldots,\alpha^{q-1}).$$

But the equality

$$\cup_{k=0}^{\infty} \operatorname{Lin}_{L_k}(1,\alpha,\ldots,\alpha^{q-1}) = \operatorname{Lin}_{\langle L^H\rangle}(1,\alpha,\ldots,\alpha^{q-1})$$

is evident, where $\langle L^H\rangle$ stands for ∂_1-differential subfield of D^H generated by L^H. The equality $D = \operatorname{Lin}_{\langle L^H\rangle}(1,\alpha,\ldots,\alpha^{q-1})$ implies $|D : \langle L^H\rangle| \le q$. But due to Artins Theorem $|D : D^H| = |H| \ge q$. These two last inequalities show that $D = \langle L^H\rangle$.

Now let us consider case $n \ge 2$. Let $'L$ stand for $(\partial_2,\ldots,\partial_n)$-subfield of D generated by L. The field $'L^H$ generates D^H as a ∂_1-field. Therefore on the one hand, as we already have shown above, the field $'L^H$ generates D^H as a ∂_1-field on the other hand by induction we can assume that L^H generates $'L^H$ as $(\partial_2, \ldots, \partial_n)$-field (because L generates $'L$ as $(\partial_2, \ldots, \partial_n)$-field). Therefore we can conclude that L^H generates D^H as $(\partial_1, \ldots, \partial_n)$-field. This is the proof of the Lemma. $\square$

Proposition 2.1. *Let D be any field and H be its any group of automorphisms. Then the inequality $|D : D^H| < \infty$ is true if and only if H is finite.*

Proof. If $|D : D^H| < \infty$ is given and $e_1, \ldots, e_m$ is any linear basis of D over D^H then every element $h \in H$ is uniquely determined by the set $\{he_i\}_{i=\overline{1,m}}$. But for every fixed i the set $\{he_i\}_{h \in H}$ is a finite set so far as for every $h \in H$ element he_i should be a root of the minimal polynomial of e_i over D^H, which has only finite number of roots. Therefore H can not have infinitely many elements i.e. H has to be finite. If H is finite then $|D : D^H| < \infty$ due to Artin's Theorem. $\square$

Now we are able to prove our main results.

Theorem 2.1. *Let $(F; \partial_1, \ldots, \partial_n)$ be any differential field and $K = \{a \in F : \partial_i a = 0$ at $i = \overline{1,n}\}$ be its constant field, l be any natural number, $x_1, \ldots, x_l$ be differential algebraic indeterminates over K, $x = (x_1, \ldots, x_l)$, H be any subgroup of $GL(l, K) \propto K^l$. Then $K(x)^H$ generates $K^\partial\langle x\rangle^H$ as a ∂-field if and only if H is a finite group.*

Proof. The field $L = K(x)$ generates $K^\partial\langle x\rangle$ as a ∂-field. Therefore if H is finite then due to the above Lemma $K(x)^H$ generates $K^\partial\langle x\rangle^H$ as such a differential field.

Vise versa, let $K(x)^H$ generate $K^\partial\langle x\rangle^H$ as a ∂-field. But as we have mentioned already $K^\partial\langle x\rangle^H \subset K^\partial\langle x\rangle$ is ∂-algebraic extension i.e.

$$\mathrm{Diff}\,.\,\mathrm{tr}\,.\,\mathrm{deg}\,.K^\partial\langle x\rangle^H/K = l.$$

So far as $K(x)^H$ generates $K^\partial\langle x\rangle^H$ as a ∂-field we can conclude that

$$\mathrm{tr}\,.\,\mathrm{deg}\,.K(x)^H/K = l = \mathrm{tr}\,.\,\mathrm{deg}\,.K(x)/K,$$

which means that $K(x)^H \subset K(x)$ is an algebraic extension. On the other hand $K(x)$ is finitely generated over K and therefore $|K(x) : K(x)^H| < \infty$. Now finiteness of H is a consequence of the Proposition 2.1. $\square$

3. $(GL^\partial(n, F), H)$-invariant differential rational functions

Let us come back to the case when the "gauge transformations" are also allowed. We consider any differential field $(F; \partial_1, \ldots, \partial_n)$ and any finite subgroup H of $GL(n + m, K) \propto K^{n+m}$. In this case $K^\partial\langle x\rangle^{GL^\partial(n,F),H)}$ has not to be invariant relative to differential operators $\partial_1, \ldots, \partial_n$ i.e $(K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}, \partial_1, \ldots, \partial_n)$ is not a differential field in common case. If $\mathrm{char} F = 0$ and the differential operators $\partial_1, \ldots, \partial_n$ are linear independent over F we are able to prove the following Theorem.

Theorem 3.1. *There are such differential operators*

$$\delta_i : K^\partial\langle x\rangle \to K^\partial\langle x\rangle$$

where $i = \overline{1,n}$, that $(K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}, \delta_1, \ldots, \delta_n)$ is a differential field and as such a differential field it is generated by $K(x)^H$.

Proof. We know that tr.deg.$K(x)^H/K = n+m$. Let $f_1(x), \dots, f_n(x)$ be any system of elements of $K(x)^H$ which is algebraic independent over K. Consider the following "column vector" δ consisting of differential operators $\delta_i : K^\delta\langle x\rangle \to K^\delta\langle x\rangle$, $i = \overline{1,n}$, given by $\delta = \phi^\partial\langle x\rangle^{-1}\partial$, where

$$\phi^\partial\langle x\rangle = (\partial f_1(x), \dots, \partial f_n(x)) = \begin{pmatrix} \partial_1 f_1(x) & \partial_1 f_2(x) & \cdots & \partial_1 f_n(x) \\ \partial_2 f_1(x) & \partial_2 f_2(x) & \cdots & \partial_2 f_n(x) \\ \vdots & \vdots & \dots & \vdots \\ \partial_n f_1(x) & \partial_n f_2(x) & \cdots & \partial_n f_n(x) \end{pmatrix}.$$

It is easy to check that

$$\phi^\partial\langle x\rangle = \begin{pmatrix} \partial_1 x_1 & \partial_1 x_2 & \cdots & \partial_1 x_{n+m} \\ \partial_2 x_1 & \partial_2 x_2 & \cdots & \partial_2 x_{n+m} \\ \vdots & \vdots & \dots & \vdots \\ \partial_n x_1 & \partial_n x_2 & \cdots & \partial_n x_{n+m} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_2(x))}{\partial x_1} & \dots & \frac{\partial f_n(x))}{\partial x_1} \\ \frac{\partial f_1(x))}{\partial x_2} & \frac{\partial f_2(x))}{\partial x_2} & \dots & \frac{\partial f_n(x))}{\partial x_2} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_1(x))}{\partial x_{n+m}} & \frac{\partial f_2(x))}{\partial x_{n+m}} & \dots & \frac{\partial f_n(x))}{\partial x_{n+m}} \end{pmatrix}$$

which shows that matrix $\phi^\partial\langle x\rangle$ is not singular so far as char$K = 0$ and the system $f_1(x), \dots, f_n(x)$ is algebraic independent over K. Moreover operators $\delta_1, \dots, \delta_n$ commute with each other because matrix $\phi^\partial\langle x\rangle$ belongs to $GL^\partial(n, K^\partial\langle x\rangle)$.

Of course operators $\delta_1, \dots, \delta_n$ "depend" on x and $\partial_1, \dots, \partial_n$. The main property of these operators is that they are invariant relative to our transformations

$$x = (x_1, \dots, x_{n+m}) \mapsto xh + h_0, \partial \mapsto g^{-1}\partial,$$

where $(h, h_0) \in H$ and $g \in GL^\partial(n, F)$, which is an easy consequence of the identity

$$\phi^{g^{-1}\partial}\langle xh + h_0\rangle = g^{-1}\phi^\partial\langle x\rangle.$$

Therefore $(K^\partial\langle x\rangle^{(GL^\partial(n,F),H)}; \delta_1, \dots, \delta_n)$ is a differential field.

The following relations are evident:

$$K^\partial\langle x\rangle^{(GL^\partial(n,F),H)} = K^\partial\langle x\rangle^{GL^\partial(n,F)} \cap K^\partial\langle x\rangle^H = \left(K^\partial\langle x\rangle^{GL^\partial(n,F)}\right)^H.$$

Moreover $K^\partial\langle x\rangle^{GL^\partial(n,F)}$ is generated by $x_1, \dots, x_{n+m}$ over K as a δ-field, that is, $K^\partial\langle x\rangle^{GL^\partial(n,F)} = K^\delta\langle x\rangle$ — the field of all δ-rational functions in $x_1, \dots, x_{n+m}$ over K. Indeed the inclusion $K^\partial\langle x\rangle^{GL^\partial(n,F)} \supset K^\delta\langle x\rangle$ is evident and if $f^\partial\langle x\rangle \in K^\partial\langle x\rangle^{GL^\partial(n,F)}$, that is,

$$f^{g^{-1}\partial}\langle x\rangle = f^\partial\langle x\rangle$$

is valid for any $g \in GL^\partial(n, F)$ then according to [3] under above assumptions this equality remains be true even at $g = \phi^\partial\langle x\rangle$. In other words the equality $f^\delta\langle x\rangle = f^\partial\langle x\rangle$ is true, which means that $f^\partial\langle x\rangle \in K^\delta\langle x\rangle$. Hence the inclusion $K^\partial\langle x\rangle^{GL^\partial(n,F)} \subset K^\delta\langle x\rangle$ is also valid. So we have equality $K^\partial\langle x\rangle^{GL^\delta(n,F)} = K^\delta\langle x\rangle$. But $K(x)$ generates $K^\delta\langle x\rangle$ as a δ-field and H is a finite group and therefore due to the above Lemma $K(x)^H$ generates $K^\delta\langle x\rangle^H = K^\partial\langle x\rangle^{(GL^\delta(n,F),H)}$ as a δ-field. This is the proof of Theorem 3.1. □

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