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Some Remarks on Generalized Inverses of Conjugate EP Matrix

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Abstract. The existence of a group inverse and characterization of generalized inverses of a Con-EP (Conjugate EP) matrix are studied and it is shown that for a Con-EP matrix A, $A^\dagger$ is not a polynomial in A and group inverse does not coincides with $A^\dagger$. Conditions are derived for A^T to be a polynomial in A for a Con-EP matrix A.

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1. Introduction

Any matrix $A \in M_n$ (the set of all $n \times n$ complex matrices) is said to be Con-EP if $R(A) = R(A^T)$ or equivalently $AA^\dagger = \overline{A^\dagger A}$ and is said to be Con-EP$_r$ if A is Con-EP and rk(A) = r [3], where $A^\dagger$, the Moore-Penrose inverse of A is the unique solution of the equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA,$$

$R(A)$ is the range space of A, $A^* = \overline{A}^T$ and rk(A) denote the rank of A.

For real matrices, the concept of Con-EP matrix coincides with EP matrix [4]. For $A \in M_n$, $X = A^\#$ is the group inverse of A satisfying $AXA = A$, $XAX = X$ and $AX = XA$ [1].

Theorem 1.1. *For a complex matrix A, if A^T is a polynomial in A with* $\mathrm{rk(A)} = \mathrm{rk(AA^T)}$*, then A is a Con-EP.*

Proof. Since A^T is polynomial in A, $AA^T = A^TA$. $N(A) \subseteq N(A^TA) = N(AA^T)$ and $\mathrm{rk(A)} = \mathrm{rk(AA^T)}$ implies $N(A) = N(AA^T)$. Also, $N(A^T) \subseteq N(AA^T) = N(A)$ and $\mathrm{rk(A^T)} = \mathrm{rk(A)}$ implies $N(A^T) = N(A)$. Thus A is Con-EP. ☐

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Remark 1.1. However, the converse of Theorem 1.1 is not true can be seen from

$$A = \begin{pmatrix} i & i \\ 0 & 1 \end{pmatrix}.$$

For this A, A^T is not a polynomial in A, A is Con-EP being nonsingular and $\mathrm{rk(A)} = \mathrm{rk(AA^T)}$.

Theorem 1.2. *For a complex matrix A, A^+ is a polynomial in $\overline{A}$, with* $\mathrm{rk(A)} = \mathrm{rk(AA^T)}$*, then A is Con-EP.*

Proof. Since A^+ is a polynomial in $\overline{A}$, $A^\dagger\overline{A} = \overline{A}A^\dagger$. $N(A^\dagger) \subseteq N(\overline{A}A^\dagger) = N(A^\dagger\overline{A})$ and $\mathrm{rk(A)} = \mathrm{rk(AA^T)}$ implies $\mathrm{rk(A^\dagger)} = \mathrm{rk(A^\dagger\overline{A})}$, hence $N(A^\dagger) = N(A^\dagger\overline{A})$. $N(\overline{A}) \subseteq N(A^\dagger\overline{A}) = N(A^\dagger)$ and $\mathrm{rk(A)} = \mathrm{rk(\overline{A})} = \mathrm{rk(A^\dagger)}$ implies $N(A) = N(A^T)$. Thus A is Con-EP. □

Remark 1.2. However, the converse of Theorem 1.2 is not true can be seen from the matrix

$$A = \begin{pmatrix} i & i \\ 0 & 1 \end{pmatrix}.$$

Remark 1.3. The condition on rank of A and AA^T is essential can be seen by the following: For

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

A^T is a polynomial in A, $A^\dagger$ is not a polynomial in $\overline{A}$, $\mathrm{rk(A)} \neq \mathrm{rk(AA^t)}$. A is not Con-EP.

2. Conjugate EP matrices and group inverses

In general for a Con-EP matrix, it's group inverse does not exist (Refer Example 2.2). The existence of the group inverse, the generalized inverses belonging to the sets $A\{1,2\}$, $A\{1,2,3\}$ and $A\{1,2,4\}$ of a Con-EP$_r$ matrix A are characterized. It is clear that, A is Con-EP$_r$ if and only if $A^\dagger$ is Con-EP$_r$. Thus the Con-EP$_r$ property of a complex matrix is preserved for it's Moore-Penrose inverse. However, other generalized inverses of a Con-EP$_r$ matrix need not be Con-EP$_r$. For instance,

$$A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$$

is Con-EP$_1$,

$$X = \begin{pmatrix} -i & 0 \\ -1 & 0 \end{pmatrix}$$

is a 1–inverse of A, which is not Con-EP$_1$.

The generalized inverses $X \in A\{1,2\}$ is shown to be Con-EP$_r$ under certain conditions in the following way.

Theorem 2.1. *Let $A \in M_n$, $X \in A\{1,2\}$ (set of all X's satisfying first two equations of $A^\dagger$) and AX, XA are Con-EP$_r$ matrices. Then A is Con-EP$_r$ if and only if X is Con-EP$_r$.*

Proof. Since AX and XA are Con-EP$_r$, $X \in A\{1,2\}$, we have $R(A) = R(AX) = R((AX)^T) = R(X^T)$ and $R(A^T) = R((XA)^T) = R(XA) = R(X)$. Now, A is Con-EP$_r$ $\Leftrightarrow$ $[R(A) = R(A^T)$ and rk(A) = r$]$ $\Leftrightarrow$ $[R(X) = R(X^T)$ and rk(A) = rk(X) = r$]$ $\Leftrightarrow$ X is Con-EP$_r$. □

Remark 2.1. In Theorem 2.1, the conditions that both AX and XA to be Con-EP$_r$ are essential. For instance,

$$A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$$

is Con-EP$_1$,

$$X = \begin{pmatrix} -i & 0 \\ -i & 0 \end{pmatrix} \in A\{1,2\}.$$

AX is Con-EP$_1$ and XA is not Con-EP$_1$. Similarly,

$$Y = \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \in A\{1,2\},$$

YA is not Con-EP$_1$ and AY is not Con-EP$_1$, X is not Con-EP$_1$, Y is not Con-EP$_1$.

Remark 2.2. For $A \in M_n$, $X \in A\{1,2\}$, if AX and XA are real symmetric matrices, then A is Con-EP$_r$ if and only if X is Con-EP$_r$. In particular, for $X = A^\dagger$ with $AA^\dagger$ and $A^\dagger A$ are real matrices it reduces to that, A is EP$_r$ if and only if A is Con-EP$_r$ if and only if $A^\dagger$ is Con-EP$_r$ if and only if $A^\dagger$ is EP$_r$.

Now we shall derive certain condition for inverses belonging to the sets $A\{1,2,3\}$ and $A\{1,2,4\}$ of a con-EPr matrix A to be Con-EP$_r$.

Theorem 2.2. *Let $A \in M_n$, $X \in A\{1,2,3\}$ (set of all X's satisfying first three equations of $A^\dagger$) and XA is EP$_r$. Then A is Con-EP$_r$ if and only if X is Con-EP$_r$.*

Proof. Since $X \in A\{1,2,3\}$ and XA is EP$_r$, $R(A) = R(AX) = R((AX)^*) = R(X^*)$ and $R(A^*) = R((XA)^*) = R(XA) = R(X) \Leftrightarrow R(A^T) = R(\overline{X})$. Now, A is Con-EP$_r$ $\Leftrightarrow$ $[R(A) = R(A^T)$ and rk(A) = r$]$ $\Leftrightarrow$ $[R(X^*) = R(\overline{X})$ and rk(A) = rk(X) = r $]$ $\Leftrightarrow$ $[R(X^T) = R(X)$ and rk(X) = r$]$ $\Leftrightarrow$ X is Con-EP$_r$. □

Remark 2.3. In Theorem 2.2, the condition that XA is EP$_r$ cannot be relaxed. For instance,

$$A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},$$

is Con-EP$_1$ matrix,

$$X = \begin{pmatrix} -i & 0 \\ -i & 0 \end{pmatrix} \in A\{1,2,3\}$$

and XA is not EP and X is not Con-EP$_1$.

Theorem 2.3. *Let $A \in M_n$, $X \in A\{1,2,4\}$ (set of X's satisfying 1,2 and 4th equations of $A^\dagger$) and AX is EP$_r$. Then, A is Con-EP$_r$ if and only if X is Con-EP$_r$.*

Proof. This can be proved along same lines as that of Theorem 2.2 and hence the proof is omitted. □

Remark 2.4. The condition that AX is EP_r can not be weakened in the Theorem 2.3. This is illustrated in the following example.

Example 2.1.

$$A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$$

is Con-EP$_1$,

$$X = \begin{pmatrix} -i & -i \\ 0 & 0 \end{pmatrix} \in \{1,2,4\}$$

is not Con-EP$_1$ and AX is not EP.

Remark 2.5. In particular for $X = A^\dagger$, since $A^\dagger \in A\{1,2,4\}$ and $AA^\dagger$ is EP$_r$ being Hermitian, then Theorem 2.3 reduces to that, A is Con-EP$_r$ if and only if $A^\dagger$ is Con-EP$_r$.

The following Theorem gives condition for the existence of $A^\#$ of a Con-EP$_r$ matrix A.

Theorem 2.4. *Let* $A \in M_n$ *be Con-EP*$_r$ *and* $\mathrm{rk}(\mathrm{A}\overline{\mathrm{A}}) = \mathrm{rk}(\mathrm{A}^2)$. *Then* $A^\#$ *exists and is Con-EP*$_r$.

Proof. Since A is Con-EP$_r$ matrix, $\mathrm{rk}(\mathrm{A}\overline{\mathrm{A}}) = \mathrm{rk}(\mathrm{A})$. By hypothesis, $\mathrm{rk}(\mathrm{A}^2) = \mathrm{rk}(\mathrm{A}\overline{\mathrm{A}}) = \mathrm{rk}(\mathrm{A})$. By [1, Theorem 2, p. 156], $A^\#$ exists for A. To show that $A^\#$ is Con-EP$_r$, it is enough to prove that $R(A^\#) = R((A^\#)^T)$. Since $AA^\# = A^\# A$, we have $R(A) = R(AA^\#) = R(A^\# A) = R(A^\#)$ and $R(A^T) = R((A^\# A)^T) = R((AA^\#)^T) = R((A^\#)^T)$. Now, A is Con-EP$_r$ $\Rightarrow$ $[R(A) = R(A^T)$ and $\mathrm{rk}(\mathrm{A}) = \mathrm{r}]$ $\Rightarrow$ $[R(A^\#) = R((A^\#)^T)$ and $\mathrm{rk}(\mathrm{A}) = \mathrm{rk}(\mathrm{A}^\#) = \mathrm{r}]$ $\Rightarrow$ $A^\#$ is Con–EP$_r$. □

Remark 2.6. In Theorem 2.4 the condition that $\mathrm{rk}(\mathrm{A}^2) = \mathrm{rk}(\mathrm{A}\overline{\mathrm{A}})$ is essential.

Example 2.2. Let

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

and $\mathrm{rk}(\mathrm{A}) = \mathrm{rk}(\mathrm{A}\overline{\mathrm{A}}) \neq \mathrm{rk}(\mathrm{A}^2)$. Now,

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = FG;$$

$GF = 0$ and hence $(GF)^{-1}$ does not exist. Therefore, $A^\# = F(GF)^{-2}G$ [1, p. 157] does not exist for a Con-EP matrix.

Bevis *et al.* [2, Theorem 5] proved that the group inverse for semi-linear transformation T on C^n induced by a matrix A exists if $R(A\overline{A}) = R(A)$. Since for a con-EP$_r$ matrix, $\mathrm{rk}(\mathrm{A}\overline{\mathrm{A}}) = \mathrm{rk}(\mathrm{A})$, the condition $R(A\overline{A}) = R(A)$ automatically holds. Hence we have following:

Theorem 2.5. *Let* $A \in M_n$ *be Con-EP*$_r$. *Then* $T^\#$ *exists for any semi-linear transformation* T *on* C^n *induced by* A *relative to the standard basis.*

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