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Some Inequalities Between Two Polygons Inscribed One in the Other

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Abstract. It is well known that, given a triangle inscribed in another triangle, the perimeters of the three external triangles can never all be simultaneously greater than the perimeter of the inscribed triangle and that furthermore they are all equal to it if and only if we put the vertices of the inscribed triangle at the midpoints of sides of the circumscribed triangle. The same result is true for the areas. The present paper shows how such a results extends to the case of two convex polygons inscribed one in other, connecting it to the classic works about inscribed and circumscribed polygons respectively with minimum and maximum perimeter.

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1. Introduction

The following neat inequality between two triangles one inscribed in the other was proposed by Debrunner [1] in 1956 and proved first by Dresel [2] in 1961 and later on, with different methods, by other mathematicians:

Theorem 1.1. *Given a triangle ABC and any other triangle $A'B'C'$ circumscribing it, it is not possible that we have simultaneously:*

$$P(ABC) < P(A'CB), P(ABC) < P(CB'A) \text{ and } P(ABC) < P(AC'B)$$

where $P(ABC)$ indicates the perimeter of triangle ABC.

Another result of the same kind, also proposed by Debrunner [3] in 1956 and proved by Bager [4] in 1957 and later on by other mathematicians, is the following:

Theorem 1.2. *Given a triangle ABC and any other triangle $A'B'C'$ circumscribing it, it is not possible that we have simultaneously:*

$$S(ABC) < S(A'CB), S(ABC) < S(CB'A) \text{ and } S(ABC) < S(AC'B)$$

where $S(ABC)$ indicates the area of triangle ABC.

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In this paper we want to show how the inequalities can be extended to the case of two convex polygons inscribed one in the other, thus achieving a result that generalizes Theorems 1.1 and 1.2.

2. The general case of convex polygons

In the case of convex polygons the following is in fact true:

Theorem 2.1. *Given a convex polygon of n sides with vertices A_i ($i = 1, \ldots, n$) numbered cyclically so that $A_0 = A_n$, and any other polygon of n sides circumscribing it with vertices A'_i (with A_i lying on the side $A'_{i-1}\ A'_i$), if we call B_i the intersection of the diagonals $A_{i-1}A_{i+1}$ and $A_i\ A_{i+2}$ (see Fig. 1), it is not possible that we have simultaneously:*

$$P(A_iB_iA_{i+1}) < P(A_iA_{i+1}A'_i) \qquad \textit{for} \qquad i = 1, \ldots, n \tag{2.1}$$

or

$$S(A_iB_iA_{i+1}) < S(A_iA_{i+1}A'_i) \qquad \textit{for} \qquad i = 1, \ldots, n\ . \tag{2.2}$$

Furthermore one and only one circumscribed n–gon exists such that $P(A_iB_iA_{i+1}) = P(A_iA_{i+1}A'_i)$ and $S(A_iB_iA_{i+1}) = S(A_iA_{i+1}A'_i)$ for $i = 1, \ldots, n$, and that is achieved by putting the side $A'_iA'_{i+1}$ parallel to the diagonal A_iA_{i+2}.

We define the angular coordinate α_i (for $i = 1, \ldots, n$) as the angle which the side $A'_{i-1}A'_i$ makes with the line parallel to diagonal $A_{i-1}A_{i+1}$ passing for vertex A_i (see Fig. 1) and fixing that the parallel line has to be rotated counterclockwise through a positive angle α_i in order to become coincident with $A'_{i-1}A'_i$.

For the law of sines we will have that:

$$\frac{\overline{A_iB_i} + \overline{B_iA_{i+1}}}{\overline{A_iA_{i+1}}} = \frac{\sin\lambda_i + \sin\mu_{i+1}}{\sin(\lambda_i + \mu_{i+1})} = \frac{2\sin\frac{\lambda_i+\mu_{i+1}}{2}\cos\frac{\lambda_i-\mu_{i+1}}{2}}{2\sin\frac{\lambda_i+\mu_{i+1}}{2}\cos\frac{\lambda_i+\mu_{i+1}}{2}} = \frac{\cos\frac{\lambda_i-\mu_{i+1}}{2}}{\cos\frac{\lambda_i+\mu_{i+1}}{2}} \tag{2.3}$$

for $i = 1, \ldots, n$ with $\lambda_i = \widehat{A_{i+2}A_iA_{i+1}}$ and $\mu_i = \widehat{A_{i-2}A_iA_{i-1}}$ (for $i = 1, \ldots, n$). Now it is evident that $\widehat{A_iA_{i+1}A'_i} = \lambda_i + \alpha_{i+1}$ and $\widehat{A_{i+1}A_iA'_i} = \mu_{i+1} - \alpha_i$ (see Fig. 1) and so that:

$$\frac{\overline{A_iA'_i} + \overline{A'_iA_{i+1}}}{\overline{A_iA_{i+1}}} = \frac{\sin(\mu_{i+1} - \alpha_i) + \sin(\lambda_i + \alpha_{i+1})}{\sin(\lambda_i + \mu_{i+1} + \alpha_{i+1} - \alpha_i)} = \frac{\cos\frac{\lambda_i-\mu_{i+1}+\alpha_{i+1}+\alpha_i}{2}}{\cos\frac{\lambda_i+\mu_{i+1}+\alpha_{i+1}-\alpha_i}{2}} \tag{2.4}$$

for $i = 1, \ldots, n$. Now, with α_i as defined above, it is evident that:

$$-\lambda_{i-1} < \alpha_i < \mu_{i+1} \qquad \text{for } i = 1, \ldots, n \tag{2.5}$$

and the denominators of last term of (2.3) and (2.4) are both positive.

Hence $\Delta_i = P(A_iB_iA_{i+1}) - P(A_iA_{i+1}A'_i)$ will have the same sign as:

$$\begin{aligned} &\cos\frac{\lambda_i - \mu_{i+1}}{2}\cos\frac{\lambda_i + \mu_{i+1} + \alpha_{i+1} - \alpha_i}{2} \\ &- \cos\frac{\lambda_i + \mu_{i+1}}{2}\cos\frac{\lambda_i - \mu_{i+1} + \alpha_{i+1} + \alpha_i}{2} \\ &= \sin\left(\lambda_i + \frac{\alpha_{i+1}}{2}\right)\sin\frac{\alpha_i}{2} - \sin\left(\mu_{i+1} - \frac{\alpha_i}{2}\right)\sin\frac{\alpha_{i+1}}{2}. \end{aligned} \tag{2.6}$$

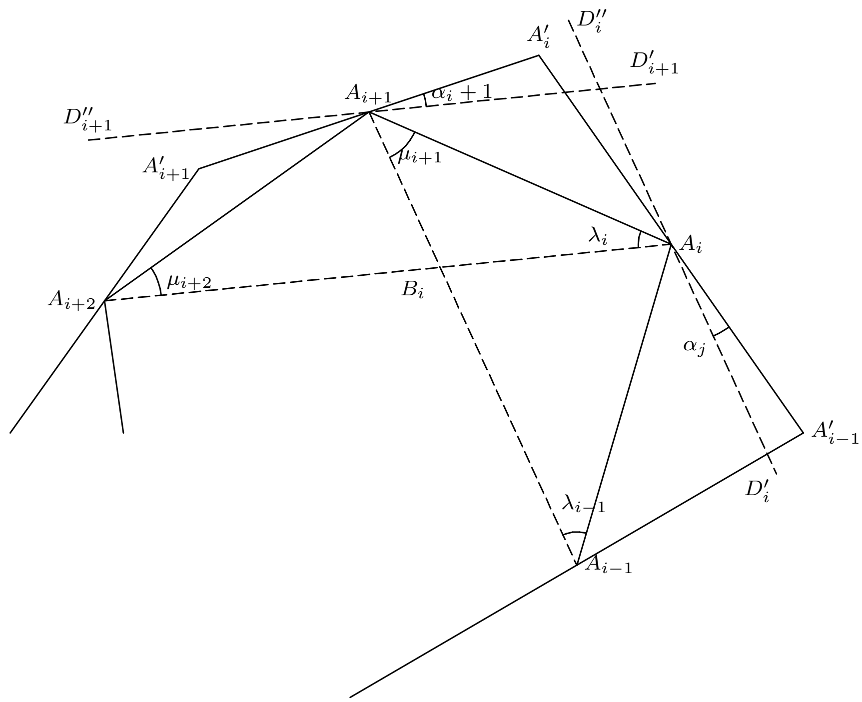

Figure 1. General case of convex polygons

There are now three possibilities:

Case (1): The α_i are all zero. Then, for the (2.6), $\Delta_i = 0$ for $i = 1, \ldots, n$ and the side $A'_i A'_{i+1}$ is always parallel to the diagonal $A_i A_{i+2}$.

Case (2): There is only one $\alpha_i \neq 0$ or there are two that don't have the same sign. In such cases, on the varying of i, there will be an i for which $\alpha_i > 0$ and $\alpha_{i+1} \leq 0$ (or $\alpha_i \geq 0$ and $\alpha_{i+1} < 0$). But for (2.5) $\mu_{i+1} - \frac{\alpha_i}{2} > 0$ and $\lambda_i + \frac{\alpha_{i+1}}{2} > 0$ and so, on the basis of (2.6), $\Delta_i \geq 0$ and the theorem is proved.

Case (3): All the α_i, for $i = 1, \ldots, n$, are different from zero and have the same sign.

In such a case, dividing (2.6) by $\sin \frac{\alpha_i}{2} \sin \frac{\alpha_{i+1}}{2}$ (> 0), Δ_i has the same sign as:

$$\begin{aligned} & \frac{\sin\left(\lambda_i + \frac{\alpha_{i+1}}{2}\right)}{\sin \frac{\alpha_{i+1}}{2}} - \frac{\sin\left(\mu_{i+1} - \frac{\alpha_i}{2}\right)}{\sin \frac{\alpha_i}{2}} \\ & = \sin \lambda_i \mathrm{cotg} \frac{\alpha_{\mathrm{i}+1}}{2} - \sin \mu_{\mathrm{i}+1} \mathrm{cotg} \frac{\alpha_{\mathrm{i}}}{2} + \cos \lambda_{\mathrm{i}} + \cos \mu_{\mathrm{i}+1} \end{aligned} \tag{2.7}$$

and supposing "ab absurdo" that $\Delta_i < 0$ for $i = 1, \dots, n$ and considering that $\cos\lambda_i + \cos\mu_{i+1} > 0$ we have:

$$\sin\lambda_i \mathrm{cotg}\,\frac{\alpha_{i+1}}{2} < \sin\mu_{i+1}\mathrm{cotg}\,\frac{\alpha_i}{2} \quad \text{for} \quad i = 1, \dots, n. \tag{2.8}$$

Now if $\alpha_i > 0$ for $i = 1, \dots, n$ (similar argument also holds if $\alpha_i < 0$) and dividing by $\sin\lambda_i \mathrm{cotg}\,\frac{\alpha_i}{2}$ (> 0) we have:

$$0 < \frac{\mathrm{cotg}\,\frac{\alpha_{i+1}}{2}}{\mathrm{cotg}\,\frac{\alpha_i}{2}} < \frac{\sin\mu_{i+1}}{\sin\lambda_i} \quad \text{for} \quad i = 1, \dots, n. \tag{2.9}$$

Multiplying (2.9) for $i = 1, \dots, n$ and remembering that

$$\prod_1^n{}_i \frac{\sin\mu_{i+1}}{\sin\lambda_i} = \prod_1^n{}_i \frac{\sin\mu_{i+1}}{\sin\lambda_{i-1}},$$

we will have:

$$\begin{aligned} 1 &< \prod_1^n{}_i \frac{\sin\mu_{i+1}}{\sin\lambda_{i-1}} \\ &= \prod_1^n{}_i \frac{\sin\frac{\lambda_{i-1}+\mu_{i+1}}{2}\cos\frac{\lambda_{i-1}-\mu_{i+1}}{2} - \cos\frac{\lambda_{i-1}+\mu_{i+1}}{2}\sin\frac{\lambda_{i-1}-\mu_{i+1}}{2}}{\sin\frac{\lambda_{i-1}+\mu_{i+1}}{2}\cos\frac{\lambda_{i-1}-\mu_{i+1}}{2} + \cos\frac{\lambda_{i-1}+\mu_{i+1}}{2}\sin\frac{\lambda_{i-1}-\mu_{i+1}}{2}} \end{aligned} \tag{2.10}$$

and dividing numerator and denominator by $\sin\frac{\lambda_{i-1}+\mu_{i+1}}{2}\cos\frac{\lambda_{i-1}-\mu_{i+1}}{2}$ (> 0) and utilising the theorem of tangents in the triangle $A_{i-1}A_iA_{i+1}$ we have:

$$\begin{aligned} 1 &< \prod_1^n{}_i \frac{1 - \mathrm{cotg}\,\frac{\lambda_{i-1}+\mu_{i+1}}{2}\,\mathrm{tg}\frac{\lambda_{i-1}-\mu_{i+1}}{2}}{1 + \mathrm{cotg}\,\frac{\lambda_{i-1}+\mu_{i+1}}{2}\,\mathrm{tg}\frac{\lambda_{i-1}-\mu_{i+1}}{2}} \\ &= \prod_1^n{}_i \frac{1 - \frac{\overline{A_iA_{i+1}} - \overline{A_{i-1}A_i}}{\overline{A_iA_{i+1}} + \overline{A_{i-1}A_i}}}{1 + \frac{\overline{A_iA_{i+1}} - \overline{A_{i-1}A_i}}{\overline{A_iA_{i+1}} + \overline{A_{i-1}A_i}}} = \prod_1^n{}_i \frac{2\overline{A_{i-1}A_i}}{2\overline{A_iA_{i+1}}} = 1 \end{aligned} \tag{2.11}$$

which is a contradiction.

To prove (2.2) we observe, first of all, that:

$$S(A_iB_iA_{i+1}) = \frac{1}{2}\overline{A_iA_{i+1}}^2 \frac{\sin\mu_{i+1}\sin\lambda_i}{\sin(\mu_{i+1}+\lambda_i)} \quad \text{for} \quad i = 1, \dots, n \tag{2.12}$$

and:

$$S(A_iA_{i+1}A'_1) = \frac{1}{2}\overline{A_iA_{i+1}}^2 \frac{\sin(\mu_{i+1}-\alpha_i)\sin(\lambda_i+\alpha_{i+1})}{\sin(\mu_{i+1}+\lambda_i+\alpha_{i+1}-\alpha_i)} \quad \text{for } i = 1, \dots, n \tag{2.13}$$

and, since $\sin(\mu_{i+1}+\lambda_i)\sin(\mu_{i+1}+\lambda_i+\alpha_{i+1}-\alpha_i) > 0$, then $\overline{\Delta}_i = S(A_iB_iA_{i+1}) + {}$ $-S(A_iA_{i+1}A'_i)$ has the same sign as:

$$\begin{aligned} &\sin\mu_{i+1}\sin\lambda_i\sin(\mu_{i+1}+\lambda_i+\alpha_{i+1}-\alpha_i) \\ &-\sin(\mu_{i+1}-\alpha_i)\sin(\lambda_i+\alpha_{i+1})\sin(\mu_{i+1}+\lambda_i) \\ &= \sin\mu_{i+1}\sin\lambda_i[\sin(\mu_{i+1}-\alpha_i)\cos(\lambda_i+\alpha_{i+1}) \\ &+\cos(\mu_{i+1}-\alpha_i)\sin(\lambda_i+\alpha_{i+1})] - \sin(\mu_{i+1}-\alpha_i)\sin(\lambda_i+\alpha_{i+1}) \times \\ &[\sin\mu_{i+1}\cos\lambda_i + \cos\mu_{i+1}\sin\lambda_i]. \end{aligned} \tag{2.14}$$

Dividing by $\sin\mu_{i+1}\sin\lambda_i\sin(\mu_{i+1}-\alpha_i)\sin(\lambda_i+\alpha_{i+1})\,(>0)$, (2.14) becomes:

$$\operatorname{cotg}(\lambda_i+\alpha_{i+1})-\operatorname{cotg}\lambda_i+\operatorname{cotg}(\mu_{i+1}-\alpha_i)-\operatorname{cotg}\ \mu_{i+1}. \tag{2.15}$$

Let's examine the three cases analysed before:

Case (1): If α_i are all zero then, from (2.15), $\overline{\Delta}_i=0$ for $i=1,\dots,n$.

Case (2): If there is an i such that $\alpha_i>0$ and $\alpha_{i+1}\le 0$ (or $\alpha_i\ge 0$ and $\alpha_{i+1}<0$) then, for (2.15), $\overline{\Delta}_i>0$ and the result is proved.

Case (3): If finally all the α_i are different from zero and with the same sign, developping (2.14), we achieve that $\overline{\Delta}_i$ has the same sign as:

$$\begin{aligned}&\sin^2\mu_{i+1}\sin\lambda_i\sin\alpha_i[\sin\lambda_i\cos\alpha_{i+1}+\cos\lambda_i\sin\alpha_{i+1}]\\&+\cos^2\mu_{i+1}\sin\lambda_i\sin\alpha_i[\sin\lambda_i\cos\alpha_{i+1}+\cos\lambda_i\sin\alpha_{i+1}]\\&-\sin\mu_{i+1}\sin^2\lambda_i\sin\alpha_{i+1}[\sin\mu_{i+1}\cos\alpha_i-\cos\mu_{i+1}\sin\alpha_i]\\&-\sin\mu_{i+1}\cos^2\lambda_i\sin\alpha_{i+1}[\sin\mu_{i+1}\cos\alpha_i-\cos\mu_{i+1}\sin\alpha_i]\end{aligned} \tag{2.16}$$

that, dividing by $\sin\alpha_i\sin\alpha_{i+1}(>0)$, become:

$$\operatorname{cotg}\alpha_{i+1}\sin^2\lambda_i-\operatorname{cotg}\alpha_i\sin^2\mu_{i+1}+\sin\mu_{i+1}\cos\mu_{i+1}+\sin\lambda_i\cos\lambda_i. \tag{2.17}$$

Now, supposing "ab absurdo" that $\overline{\Delta}_i<0$ for $i=1,\dots,n$ and considering that $\sin\mu_{i+1}\cos\mu_{i+1}+\sin\lambda_i\cos\lambda_i>0$, we will have that:

$$\operatorname{cotg}\alpha_{i+1}\sin^2\lambda_i<\operatorname{cotg}\alpha_i\sin^2\mu_{i+1}\quad\text{for}\quad i=1,\dots,n \tag{2.18}$$

and, supposing $\alpha_i>0$ for $i=1,\dots,n$ (if $\alpha_i<0$ similar argument holds but reversing the inequality), we have:

$$0<\frac{\operatorname{cotg}\alpha_{i+1}}{\operatorname{cotg}\alpha_i}<\frac{\sin^2\mu_{i+1}}{\sin^2\lambda_i}\quad\text{for}\quad i=1,\dots,n \tag{2.19}$$

and, multiplying (2.19) for $i=1,\dots,n$, we will have, for the sine rule, that:

$$1<\prod_1^n{}_i\frac{\sin^2\mu_{i+1}}{\sin^2\lambda_{i-1}}=\prod_1^n{}_i\frac{\overline{A_{i-1}A_i}^2}{\overline{A_iA_{i+1}}^2}=1 \tag{2.20}$$

that is again a contradiction.

3. Conclusive considerations

It is clear that if $n=3$ the minimum diagonals $A_{i-1}A_{i+1}$ coincide with the sides of triangle $A_1A_2A_3$, the points B_i with the opposite vertices and Theorem 2.1 is reduced to Theorems 1.1 and 1.2; Theorem 2.1 being thus a generalization of Theorems 1.1 and 1.2.

Another way to express Theorem 2.1 is the following. Taking the convex polygon with vertices $A_i (i=1,\dots,n)$ we consider the ellipses which have foci in two subsequent vertices A_i and A_{i+1} and which pass through the point of intersection B_i. If we now imagine throwing a ball from one of the vertices, towards the outside and beyond the external angle opposite the vertex of the angle of the polygon, and we bounce it subsequently on the n ellipses we will see that it will return to the initial vertex and that the direction of departure and that of arrival, seen from the inside of polygon, will always form an angle $\ge\pi$.

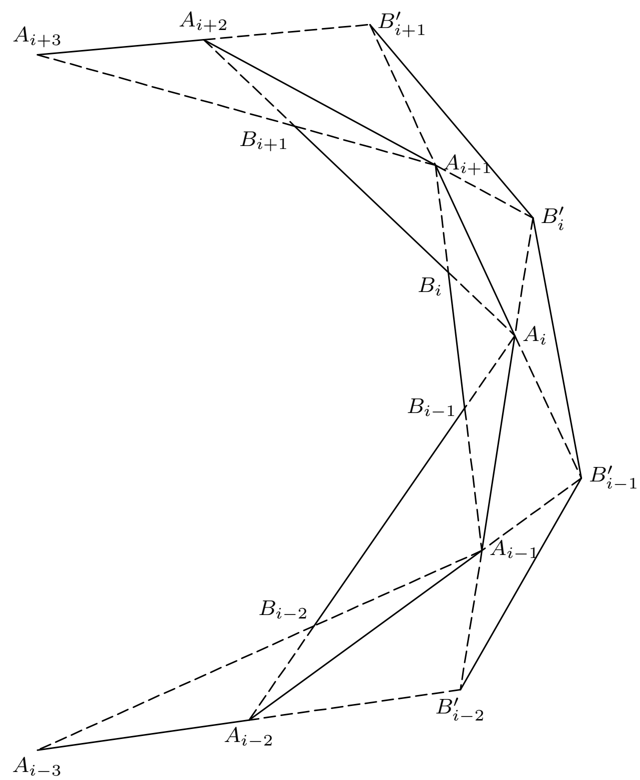

Figure 2. Internal and external polygons

It is interesting to remark too that the points B_i also form a convex n–gon which corresponds, from a projective point of view, to the dual n–gon B'_i made by intersections of sides $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$ of polygon A_i adjacent to the same side (see Fig. 2) and that to Theorem 2.1: $\exists i\, P(A_iB_iA_{i+1}) \geq P(A_iA_{i+1}A'_i)$ corresponds to the dual result $\forall\, i\, P(A_iB'_iA_{i+1}) > P(A_iA_{i+1}A'_i)$ and likewise for areas.

In substance the result achieved tell us that the n vertices of circumscribed polygon A'_i cannot simultaneously (in a manner of speaking) "distance themselves" from the corresponding sides of the inscribed polygon A_iA_{i+1} further than the corresponding points B_i are "distant" from that side.

The result is in fact, in a manner of speaking, connected to the classic works about minimum and maximum perimeters of respectively inscribed and circumscribed polygons represented by the well known (see e.g. [5]).

Theorem 3.1. *The n–gon with vertices A_i inscribed in a given n–gon with vertices $A'_i\,(i = 1,\ldots,n)$ with minimum perimeter in all it is that in which $\mu_{i+1} - \alpha_i = \lambda_{i-1} + \alpha_i$ for $i = 1,\ldots,n$.*

Using the notations introduced before we can say that the arithmetic mean of Δ_i is maximum for $\alpha_i = \frac{\mu_{i+1}-\lambda_{i-1}}{2}$ for $i = 1,\ldots,n$ and that instead, in order to maximize the minimum of Δ_i and of $\overline{\Delta}_i$, the following will be true $\alpha_i = 0$ for $i = 1, \ldots, n$ and in such a case will be $\min\{\Delta_i | i = 1, \ldots, n\} = \min\{\overline{\Delta}_i | i = 1, \ldots, n\} = 0$.

Of course finally if the n–gon A_i' is regular the two inscribed n–gons will coincide.

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