Submanifolds of a Lorentzian Para-Sasakian Manifold

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Abstract. Recently, Matsumoto [1] introduced the notion of Lorentzian para-contact structure and studied its several properties. The object of the present paper is to study the submanifolds of Lorentzian para-Sasakian manifolds.

2000 Mathematics Subject Classification: 53C25, 53C40

Key words and phrases: Lorentzian Para-Sasakian manifold, totally umbilical submanifold, anti-invariant distribution.

1. Introduction

Let $\bar{M}$ be a real $n$-dimensional manifold of class $C^\infty$ endowed with an endomorphism $\phi$ of the tangent bundle, a $C^\infty$-vector field $\xi$ which is called the structure vector field, a 1-form $\eta$ and a Lorentzian metric $g$ with signature $(-, +, +, +)$ satisfy

\begin{align}
\phi^2 X &= X + \eta(X)\xi, \quad \eta(\xi) = -1, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X)
\end{align}

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian para-contact structure and the manifold $\bar{M}$ with a Lorentzian para-contact structure is called a Lorentzian para-contact manifold [1].

Also, in a Lorentzian para-contact structure the following relations hold:

\begin{align}
\phi \xi &= 0, \quad \eta \circ \phi = 0, \quad \text{rank}(\phi) = n - 1.
\end{align}

A Lorentzian para-contact manifold $M$ is called a Lorentzian para-Sasakian (LP-Sasakian) manifold if

\begin{align}
(\nabla_X \phi)(Y) &= g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \\
\nabla_X \xi &= \phi X
\end{align}

Received: November 22, 2004; Revised: January 4, 2005.
for any \( X, Y \in T\bar{M} \), where \( \nabla \) is the Riemannian connection with respect to \( g \). Again, if we put \( \Omega(X, Y) = g(X, \phi Y) \), then \( \Omega \) is a symmetric \((0,2)\) tensor field \([1]\). Thus we have from (1.5)

\[
\Omega(X, Y) = (\nabla_X \eta)Y.
\]

Also, from (1.4), it follows that

\[
(\nabla_Z \Omega)(X, Y) = g(Y, Z)\eta(X) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)
\]

for any \( X, Y, Z \in T\bar{M} \).

LP-Sasakian manifolds have also been studied by K. Matsumoto and Mihai \([2]\), Mihai and Rosca \([3]\) and Matsumoto, De and Shaikh \([4]\). Let \( M \) be a Riemannian submanifold of a semi-Riemannian manifold \( \bar{M} \). Then the Gauss and Weingarten formulae are given by

\[
\nabla_X Y = \nabla_X Y + h(X, Y),
\]

\[
\nabla_X N = -A_N X + \nabla_X N
\]

for each \( X, Y \in TM \) and each \( N \in T^\perp M \), where \( \nabla \) is the Levi-Civita connection on \( M \), \( \nabla^\perp \) is the normal connection on the normal bundle \( T^\perp M \), \( h \) is the second fundamental form of \( M \) and \( A_N \) is the shape operator with respect to the normal section \( N \). Then we know that

\[
g(h(X, Y), N) = g(A_N X, Y)
\]

for each \( X, Y \in TM \) and \( N \in T^\perp M \). We denote by the same symbol \( g \) both metrics on \( \bar{M} \) and \( M \).

**Definition 1.1.** A submanifold \( M \) is said to be

(i) totally geodesic in \( \bar{M} \) if

\[
h = 0 \quad \text{or equivalently} \quad A_N = 0
\]

for any \( N \in T^\perp M \).

(ii) minimal in \( \bar{M} \) if the mean curvature vector \( H \) satisfies

\[
H^{\text{def}} \frac{\text{Tr}(h)}{\text{dim} \bar{M}} = 0
\]

and (iii) totally umbilical if

\[
h(X, Y) = g(X, Y)H.
\]

In Section 2 of this paper, we obtain some properties of submanifolds of an LP-Sasakian manifold. In the last section, we prove the main result concerning the non-existence of an anti-invariant distribution on the submanifold of an LP-Sasakian manifold which include the results of \([5]\) as particular case. Sub-manifolds of an LP-Sasakian manifold have been studied by two of the present authors \([5]\), Prasad \([6]\), Kalpana and Guha \([7]\) and others. Throughout this paper, we shall assume the following notation:
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(a) $M$ is a submanifold of an LP-Sasakian manifold $\bar{M}$.
(b) \{\xi\} is the 1- dimensional distribution spanned by $\xi$.
(c) $TM$ and $T^\perp M$ are the tangent and normal bundles of $M$, respectively, and
(d) $D^\perp$ is an anti- invariant distribution (i.e. $\phi D^\perp \subset T^\perp M$) of $M$ such that $D^\perp \cap \{\xi\} = 0$.

2. Submanifolds of an LP-Sasakian manifold

We first prove the following lemma:

**Lemma 2.1.** For a submanifold $M$ of an LP-Sasakian manifold $\bar{M}$, we have

\[
\phi X = \nabla_X \xi + h(X, \xi), \quad \xi \in TM,
\]
\[
\phi X = -A_\xi X + \nabla_X^\perp \xi, \quad \xi \in T^\perp M,
\]
\[
\eta(A_N X) = 0, \quad \xi \in T^\perp M,
\]
\[
\eta(A_N X) = g(\phi X, N), \quad \xi \in TM
\]

for $X \in TM$ and $N \in T^\perp M$.

**Proof.** From (1.5) and (1.9), we get (2.1). Also, from (1.5) and (1.10), we obtain (2.2). Again, in view of (1.2), (2.3) is obvious. Lastly, for $\xi \in TM$, we get

\[
\eta(A_N X) = g(\xi, A_N X) = -g(\xi, \nabla_X N) = g(\nabla_X \xi, N) = g(\phi X, N),
\]

where (1.2), (1.5) and (1.10) have been used. These complete the proof of our lemma.

**Theorem 2.1.** Let $M$ be a submanifold of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field $\xi$ is tangent to $M$. Then $M$ is invariant (i.e. $\phi TM \subset TM$) if and only if $h(X, \xi) = 0$, and $M$ is anti-invariant (i.e. $\phi TM \subset T^\perp M$) if and only if $\nabla_X \xi = 0$.

Since it is trivial from (2.1), we omit to prove our theorem.

**Theorem 2.2.** If $M$ is a totally umbilical submanifold of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field $\xi$ is tangent to $M$, then

(i) $M$ is necessarily minimal and consequently totally geodesic and
(ii) $M$ is an invariant submanifold of $\bar{M}$ and $\nabla_X \xi \neq 0$.

**Proof.** Let $M$ be totally umbilical. Using (1.2), (1.3) and (2.1) in (1.14), we get

\[
0 = h(\xi, \xi) = g(\xi, \xi)H = -H.
\]

Hence, in view of (1.13) and (1.14), we obtain (i). The second part follows from Theorem 2.1 and the above (i).

**Theorem 2.3.** A submanifold $M$ of an LP-Sasakian manifold $\bar{M}$ such that the structure vector field $\xi$ is normal to $M$ is anti-invariant in $\bar{M}$ if and only if $A_\xi X = 0$. Consequently, if $M$ is totally geodesic, then it is anti-invariant.
Proof. Since $\xi$ is normal to $M$, by virtue of (1.10) and (2.2), we get
\[ g(\phi X, Y) = -g(A_\xi X, Y) = -g(h(X, Y), \xi), \quad X, Y \in TM, \]
which provides the proof of our theorem. \(\square\)

3. Non-existence of an anti-invariant distribution

First, we prove the following:

Lemma 3.1. For a submanifold $M$ of an LP-Sasakian manifold $\bar{M}$, we have

(3.1) \[ (\nabla_Z \Omega)(X, Y) = -g(A_{\phi Y} X, Z) - \Omega(X, \nabla_Z Y) - \Omega(X, h(Z, Y)) \]
for $Y \in D^\perp$, $X, Z \in TM$.

(3.2) \[ (\nabla_Z \phi)(X) = -A_{\phi Y} Z + \nabla_Z \phi Y - \phi(\nabla_Z Y) \]
for $X, Y \in D^\perp$, $Z \in TM$.

Proof. Let $Y \in D^\perp$, $Z \in TM$. Then, by virtue of (1.11) and the fact $\phi Y \in T^\perp M$, we get
\[ (\nabla_Z \phi Y = -A_{\phi Y} Z + \nabla_Z \phi Y - \phi(\nabla_Z Y). \]
Using this equation in (1.7), we can easily derive (3.1). \(\square\)

Next, in the special case of $X \in D^\perp$, since $\phi X \in T^\perp M$, (3.1) in view of (1.5) and (1.11) yields (3.2).

Lemma 3.2. Let $M$ be a submanifold of an LP-Sasakian manifold $\bar{M}$ and $D^\perp \perp \{\xi\}$. Then we get

(3.4) \[ (\nabla_Z \Omega)(X, X) = 0 \]
for $X \in D^\perp$ and $Z \in TM$. And consequently

(3.5) \[ A_{\phi X} X = 0 \]
for $X \in D^\perp$.

Proof. Since $D^\perp \perp \{\xi\}$, we have $\eta(X) = 0$ for any $X \in D^\perp$ and hence in view of (1.8), we get (3.4). Again (3.5) follows from (3.2) and (3.4). \(\square\)

Theorem 3.1. There does not exist any anti-invariant distribution $D^\perp$ on a submanifold $M$ of an LP-Sasakian manifold $\bar{M}$ if $\xi$ is tangent to $M$ and $D^\perp \perp \{\xi\}$.

Proof. Since $D^\perp \perp \{\xi\}$, we get $\eta(X) = 0$ for any $X \in D^\perp$. Thus, from (1.2), (2.4) and (3.5), we have
\[ 0 = \eta(A_{\phi X} X) = g(\phi X, \phi X) = g(X, X) \]
for any $X \in D^\perp$. This means $D^\perp = \{0\}$. This proves the theorem. \(\square\)
A submanifold $M$ of an LP-Sasakian manifold $\bar{M}$ is said to be semi-invariant submanifold [8] if the following conditions are satisfied

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where $D$, $D^\perp$ are orthogonal differentiable distributions on $M$ and $\{\xi\}$ is the 1-dimensional distribution spanned by $\xi$,

(ii) The distribution $D$ is invariant by $\phi$, that is, $\phi D_x = D_x$ for each $x \in M$,

(iii) The distribution $D^\perp$ is anti-invariant under $\phi$, that is $\phi D^\perp \subset T^\perp_x M$ for each $x \in M$.

If both the distribution $D$ and $D^\perp$ are non-zero then the semi-invariant submanifold is called a proper semi-invariant submanifold.

Hence by virtue of Theorem 3.1, we have the following:

**Corollary 3.1.** An LP-Sasakian manifold does not admit any proper semi-invariant submanifold.

**Acknowledgment.** The authors are thankful to the referee for his valuable suggestions in the improvement of the paper.

**References**


