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Strongly ψ-Bounded and Classes of Linear Operators in Probabilistic Normed Space

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Abstract. In this paper we redefined the definition of a bounded linear operator in probabilistic normed space by introducing the notion of strongly ψ-bounded linear maps. We then show that this new definition of boundedness implies all contraction functions in probabilistic normed space are bounded. Also, we introduce the classes of linear operators in probabilistic normed space, as the set of all certainly bounded $L_c(V, V')$, D-bounded $L_D(V, V')$, strongly B-bounded $L_B(V, V')$, and strongly ψ-bounded $L_\psi(V, V')$ we then prove they are linear spaces.

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1. Introduction and preliminaries

In 1942 Menger introduced the notion of probabilistic metric space as a natural generalization of the notion of a metric space; specifically, he looked at the distance concept as a probabilistic rather than a deterministic notion. More precisely, instead of associating a number − the distance $d(p, q)$ − for every pair of elements p, q one should associate a distribution function F_{pq} and, for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x.

In complete analogy with the classical case, we then have the notion of a probabilistic normed space. This was introduced by A. N. Serstnev in 1963 and later improved by Alsina *et al.* in 1993 [2]. It is this latter definition of a probabilistic normed space which is of interest to us. In fact, we are interested to apply the notion of boundedness. A rather thorough study on this notion was, of course, done by Lafuerza Guillen *et al.* [7] and Jebril & Ali [6].

This motivates us to look for a better definition of boundedness. In this paper, we introduce the new notion of strongly ψ-bounded linear maps and we show that all of the (probabilistic) contraction functions found in the literature are bounded with respect to this definition (see section 2).

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Before we proceed we must state some definitions, known facts, and, technical results to be used in the sequel, the concepts used are those of [6] and [13]: The space of probability distribution functions (briefly, a d.f.) which we will consider are

$$\Delta^+ = \{F : [0,\infty] \to [0,1] | F \text{ is left-continuous, non-decreasing,} \\ F(0) = 0 \text{ and } F(+\infty) = 1\},$$

and the set of all F in Δ^+ for which $l^- F(+\infty) = F(+\infty) = 1$ by D^+.

In particular for any $a \geq 0$, ε_a is the d.f defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, viz. for all $F, G, H \in \Delta^+$, we have

$$\tau(\tau(F,G),H) = \tau(F,\tau(G,H)),$$

$$\tau(F,G) = \tau(G,F),$$

$$\tau(F,H) \leq \tau(G,H) \text{ if } \quad F \leq G,$$

$$\tau(F,\varepsilon_0) = F,$$

continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

A *continuous t-norm T* is, a continuous binary operation on [0,1] that is associative, commutative, nondecreasing and has 1 as identity; T^* will denote a continuous *t-conorm*, namely a continuous binary operation on [0,1] that is related to the continuous t-norm T through

$$T^*(x,y) = 1 - T(1-x, 1-y).$$

Definition 1.1. *A* probabilistic normed *space is a quadruple (V, v, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions, and v is a mapping from V into Δ^+ such that, for all p, q in V, the following conditions hold:*

(PN1) $v_p = \varepsilon_0$ if and only if $p = \theta$, θ being the null vector in V;
(PN2) $\nu_{-p} = \nu_p$;
PN3) $v_{p+q} \geq \tau(\nu_p, \nu_q)$;
(PN4) $\nu_p \leq \tau^*(\nu_{\alpha p}, v_{(1-\alpha)p})$, for all α in [0,1].

If, instead of (PN1), we only have $v_\theta = \varepsilon_0$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (PN4) is replaced by the equality $\nu_P = \tau_M(\nu_{\alpha p}, v_{(1-\alpha)p})$, then the PN space is called a *Serstnev space.*

The pair (V, v) is said to be a *Probabilistic Seminormed Space* (briefly a PSN space) if $v : V \to \Delta^+$ satisfies (PN1) and (PN2).

There is a (ε, λ)-*topology* in the PN space (V, v, τ, τ^*) which is generated by the family of neighborhoods, N_p of $p \in V$ in the following way:

$$\mathrm{N}_p(\varepsilon, \lambda) = \{N_p(\varepsilon, \lambda)\}_{\varepsilon>0,\lambda\in(0,1)}\ ,\ N_p(\varepsilon, \lambda) = \{q \in V : v_{q-p}(\varepsilon) > 1 - \lambda\},$$

and the strong topology in the PN space (V, v, τ, τ^*), is defined by the neighborhoods

$$\mathbf{U}_p(\delta) = \{U_p(\delta)\}_{\delta>0}\ ,\ U_p(\delta) = \{q \in V : v_{q-p}(\delta) > 1 - \delta\}.$$

The strong neighborhood system is equivalent to (ε, λ)-neighborhoods. Hence the strong topology coincides with the (ε, λ)-topology generated by the (ε, λ)-neighborhoods (see [13]).

We shall also need the following lemma which is due to Lafuerza Guillen *et al.* [8].

Lemma 1.1. *If $f : (V, v, \tau, \tau^*) \to (\Re, \mu, \sigma, \sigma^*)$, α is not a positive integer, and A is D-bounded, there is D-bounded, there is $n \in Z$, such that $n - 1 < \alpha < n$, for every $p \in A$ one has*

$$\mu_{\alpha f_p} \geq \mu_{n f_p}.$$

Now we are going to review the most important contraction functions in PN space.

Definition 1.2. *Let (V, v, τ, τ^*) be PN spaces. A mapping $f : V \to V$ is called a:*

(i) *C-contraction [4], if there exists $k \in (0, 1)$ such that for all $p \in V$ the following implication holds:*

$$x \in (0, \infty), \quad v_p(x) > 1 - x \Rightarrow v_{fp}(kx) > 1 - kx.$$

(ii) *weak C-contraction [9] (shortly $w - C$ contraction), if there exists $k \in (0, 1)$ such that for all $p \in V$ the following implication holds:*

$$x \in (0, 1), \quad v_p(x) > 1 - x \Rightarrow v_{fp}(kx) > 1 - kx.$$

(iii) *$(\varepsilon - \lambda)$-probabilistic contraction [5], if there exists $k \in (0, 1)$ such that $\varepsilon > 0, \lambda \in (0, 1)$ and for all $p \in V$ the following implication holds:*

$$v_p(\varepsilon) > 1 - \lambda \Rightarrow v_{fp}(k\varepsilon) > 1 - k\lambda.$$

(iv) *(k_1, k_2)-contraction, of $(\varepsilon - \lambda)$-type $(k_1, k_2 \in (0, 1))$[9], if $\varepsilon > 0$, $\lambda \in (0, 1)$ and for all $p \in V$ the following implication holds:*

$$v_p(\varepsilon) > 1 - \lambda \Rightarrow v_{fp}(k_1\varepsilon) > 1 - k_2\lambda.$$

(v) *B-contraction [13], if there exists a $k \in (0, 1)$ such that for every $p \in V$ and every $x > 0$:*

$$v_{fp}(kx) \geq v_p(x).$$

(vi) *strict B-contraction [12], if, for some $k \in (0,1)$,*

$$v_{fp}(kx) \geq \frac{v_p(x)}{v_p(x) + k(1 - v_p(x))} \quad , \quad \forall p \in V, \forall x > 0.$$

(vii) *probabilistic contraction of type $r \in (0,1)$ [11], if there exists $k \in (0,1)$ such that:*

$$v_{fp}(k^r x) \geq \frac{v_p(x)}{v_p(x) + k^{1-r}(1 - v_p(x))} \ , \forall p \in V, \forall x > 0.$$

2. Bounded contraction functions

The definition of bounded sets in a probabilistic normed space was defined in [7].

Definition 2.1. *Let (V, v, τ, τ^*) be a PN space and A be the nonempty subset of V. The probabilistic radius of A is the function R_A defined on $\Re^+$ by*

$$R_A(x) = \begin{cases} l^- \inf\limits_{p \in A} v_p(x), & x \in [0, +\infty); \\ 1, & x \in +\infty. \end{cases}$$

where $l^- f(x)$ denotes the left limit of the function f at the point x.

Definition 2.2. *A nonempty set A in a PN space (V, v, τ, τ^*) is said to be:*

(a) *Certainly bounded, if $R_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$;*
(b) *Perhaps bounded, if one has $R_A(x) < 1$ for every $x \in (0, +\infty)$ and $l^- R_A(+\infty) = 1$;*
(c) *Perhaps unbounded, if $R_A(x_0) > 0$ for some $x_0 \in (0, +\infty)$ and $l^- R_A(+\infty) \in (0, 1)$;*
(d) *Certainly unbounded, if $l^- R_A(+\infty) = 0$, i.e., if $R_A = \varepsilon_\infty$.*

Moreover, A will be said to be *distributionally bounded*, or simply *D-bounded* if either (a) or (b) holds, i.e., if $R_A \in D^+$; otherwise, i.e., if $R_A \in \Delta^+ \backslash D^+$, A is said to be *D-unbounded.* Note that in the previous definition we have used $\inf\limits_{p \in A} v_p(x_0)$ instead of R_A (see Lafuerza-Guillen *et al.* [7]).

The definition of a bounded linear operator in PN space was previously studied by Lafuerza Guillen *et al.* [7].

Definition 2.3. *Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $f : V \to V'$ is said to be* bounded *if it satisfies either one of the following conditions.*

(1) Certainly bounded*: if every certainly bounded set A of the space (V, v, τ, τ^*) has, as image by f a certainly bounded set fA of the space $(V', \mu, \sigma, \sigma^*)$, i.e., if there exists $x_0 \in (0, +\infty)$ such that $v_P(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{fp}(x_1) = 1$ for all $p \in A$.*
(2) D-Bounded*: if it maps every D-bounded set of V into a D-bounded set of V', i.e., if, and only if, it satisfies the implication, where $\varphi_A(x) = \inf\limits_{p \in A} v_p(x_0)$,*

$$\lim_{x \to +\infty} \varphi_A(x) = 1 \Rightarrow \lim_{x \to +\infty} \varphi_{fA}(x) = 1,$$

for every nonempty subset A of V.

(3) Strongly B-bounded*: if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$, $\mu_{fp}(x) \geq v_p({}^{x}\!/_{k})$, or equivalently if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,*

$$\mu_{fp}(hx) \geq v_p(x).$$

We remark this definition enables to conclude the boundedness of one kind of contraction functions in PN space (Definition 1.3(v)). Also I. Jebril and R. Ali [6] used notions (a), (b), (c) together with the following condition:

(4) Strongly C-bounded*: if there exists a constant $h \in (0, 1)$ such that, for every $p \in V$ and for every $x > 0$,*

$$v_p(x) > 1 - x \Rightarrow \mu_{fp}(hx) > 1 - hx.$$

Also, this definition enables to conclude the boundedness of three kinds of contraction functions in PN space (see Definition 1.3 (i), (ii) and (iii)). Nevertheless, it fails to include most of the (probabilistic) contraction functions.

Now we are going to introduce a definition that enables us to conclude the boundedness of all kinds of contraction function in PN space.

Definition 2.4. *Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $f : V \to V'$ is said to be bounded if it satisfies either one of (a), (b), (c) or the following condition.*

(d) Strongly $\psi-$bounded*: if there exists a $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(x) < x$, for all $x > 0$ so that the following implication holds for every $p \in V$ for every $x > 0$:*

$$v_p(x) > 1 - x \Rightarrow \mu_{fp}(\psi(x)) > 1 - \psi(x).$$

The main idea of a new definition it is used the notions of strongly $\psi-$*bounded,* this notions is generalize the condition (f) where, $\psi(x) = hx, \forall h \in (0, 1)\ \&x > 0$.

In [6] and [7] the authors show that the notions of strongly C-bounded, strongly B-bounded, D-bounded and certainly bounded do not imply each other. Then in order to show the relation between (a), (b), (c), and (f) it is enough to introduce an example about a mapping which is strongly B-bounded operator but which is not strongly $\psi-$bounded also the converse need not be true.

Example 2.1. Let $V = V' = R$ and $v_0 = \mu_0 = \varepsilon_0$, while, if $p \neq 0$, then, for $x > 0$, let $v_p(x) = G(\frac{x}{|p|})$, $\mu_p(x) = U(\frac{x}{|p|})$, where

$$G(x) = \begin{cases} \frac{9}{10}, & 0 < x \leq 1, \\ 1, & 1 < x \leq \infty, \end{cases} ,$$

$$U(x) = \begin{cases} \frac{1}{10}, & 0 < x \leq 1, \\ 1, & 1 < x \leq \infty, \end{cases}$$

Consider now the identity map $I : (R, |\ .\ |, G, v) \to (R, |\ .\ |, U, \mu)$.

i. I is strongly B-bounded operator, such that for every $p \in R$ and every $x > 0$, let $k > \max\{\frac{|p|}{x}, 1\}$ then

$$\mu_{Ip}(kx) = \mu_p(kx) = U(\frac{kx}{|p|}) = 1 = G(\frac{kx}{|p|}) = v_p(kx) > v_p(x).$$

ii. I is not strongly ψ-bounded, such that for every mapping $\psi(x) < x \;\forall\; x > 0$.

Let $p \in (x, 9/10), x \in (1/10, 8/10)$, the condition $v_p(x) > 1 - x$ is satisfied, but we note that

$$\mu_{Ip}(\psi(x)) = U(\tfrac{\psi(x)}{|p|}) \leq U(\tfrac{x}{|p|}) = \tfrac{1}{10} < 1 - x < 1 - \psi(x).$$

Now we are going to show that boundedness as defined in 2.5 will include all the probabilistic contraction functions available in the literature.

All mappings from the previous Definition 1.3 are bounded linear operators as the following argument show:

(1) If $\psi(x) = kx, k \in (0, 1)$ then (i), (ii) and (iii) are strongly ψ-bounded.

(2) If f is a (k_1, k_2) contraction of (ε, λ)-type for $k = \max\{k_1, k_2\}$ we obtain that

$$v_{fp}(k_1\varepsilon) > 1 - k_2\lambda \Rightarrow v_{fp}(k\varepsilon) > 1 - k_2\lambda \Rightarrow v_{fp}(k\varepsilon) > 1 - k\lambda \;,$$

thus the class of (k_1, k_2) contractions of (ε, λ)-type coincides with that of (ε, λ)-probabilistic contractions.

(3) Every B-contraction functions is strongly B-bounded.

(4) Let us consider the mapping

$$\psi(x) = \tfrac{kx}{1-x+kx}, \quad k \in (h, 1), \text{ where } h = \max\{0, x-1/x\}.$$

In addition, we can easily see that $\psi(x) < x, \forall x \in (0, 1)$ and $\psi(x) > kx$, $\forall x > 0$ and that

$$v_p(x) > 1 - x \Leftrightarrow \frac{v_p(x)}{v_p(x) + k(1 - v_p(x))} > 1 - \Psi(x).$$

Consequently, if f is a strict B-contraction then

$$\begin{aligned} v_p(x) > 1 - x &\Rightarrow \tfrac{v_p(x)}{v_p(x)+k(1-v_p(x))} > 1 - \psi(x) \\ &\Rightarrow v_{fp}(kx) > 1 - \psi(x) \\ &\Rightarrow v_{fp}(\psi(x)) > 1 - \psi(x). \end{aligned}$$

Thus every strict B-contraction is strongly ψ-bounded.

(5) If f is a probabilistic contraction of type $r \in (0, 1)$ and $\lambda = \max\{k^r, k^{1-r}\}$ then

$$v_{fp}(\lambda x) \geq v_{fp}(k^r x) \geq \tfrac{v_p(x)}{v_p(x)+k^{1-r}(1-v_p(x))} \geq \tfrac{v_p(x)}{v_p(x)+\lambda(1-v_p(x))} \quad .$$

Thus, every probabilistic contraction of type $r \in (0, 1)$ is a strict B-contraction, so it is also strongly ψ-bounded.

We recall the following theorem from [7].

Theorem 2.1. *Every strongly B-bounded linear operator f is continuous with respect to the strong topologies in (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$, respectively.*

In the following theorem we show that every strongly ψ-bounded linear operator f is continuous.

Theorem 2.2. *Every strongly ψ-bounded linear operator f is continuous.*

Proof. Due to Corollary 3.1[7], it suffices to verify that f is continuous at θ, the null vector in V. Let $N_{\theta'}(\varepsilon)$, with $\varepsilon > 0$, be an arbitrary neighborhood of θ' and $\lambda \in (0,1)$ are given, we choose δ such that $0 < \psi(\delta) < \min\{\varepsilon, \lambda\}$, then:

$$p \in N_\theta(\delta) \Leftrightarrow v_p(\delta) > 1-\delta \Rightarrow \mu_{fp}(\psi(\delta)) > 1-\psi(\delta)$$
$$\Rightarrow \mu_{fp}(\varepsilon) > 1-\psi(\delta) > 1-\lambda,$$

that is $f_p \in N_{\theta'}(\varepsilon, \lambda)$; in other words, f is continuous. □

3. Classes of linear operators in PN space

Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces, and let $L(V, V')$ be the vector space of linear operators $f : V \ \rightarrow V'$.

The following is our definition of some classes in PN space.

Definition 3.1. *Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. Then*

(1) $L_c(V, V')$ *is called a* certainly bounded subset, *where* $L_c(V, V') = \{f : V \rightarrow V', f$ *is a* certainly bounded linear *operators* $\}$.
(2) $L_D(V, V')$ *is called a* D-bounded subset*, where* $L_D(V, V') = \{f : V \rightarrow V', f$ *is a D-bounded linear operators*$\}$.
(3) $L_B(V, V')$ *is called a* B-bounded subset*, where* $L_B(V, V') = \{f : V \rightarrow V', f$ *is a strongly B-bounded linear operators* $\}$.
(4) $L_\psi(V, V')$ *is called a* ψ-bounded subset*, where* $L_\psi(V, V') = \{f : V \rightarrow V', f$ *is a strongly ψ-bounded linear operators*$\}$.

After Definition 3.1, we are going to prove that $L_c(V, V')$, $L_D(V, V')$, $L_B(V, V')$ and $L_\psi(V, V')$ are linear spaces.

Theorem 3.1. *$L_c(V, V')$ is a linear space.*

Proof. Let f and g be two certainly bounded linear operators from (V, v, τ, τ^*) into $(V, \mu, \sigma, \sigma^*)$ and let A be certainly bounded subset of V. By Definition 2.4, we note that if there exists $x_0 \in (0, +\infty)$ such that $v_P(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{fp}(x_1) = 1$ for all $p \in A$ and if there exists $x_2 \in (0, +\infty)$ such that $v_P(x_2) = 1$ for all $p \in A$, then there exists $x_3 \in (0, +\infty)$ such that $\mu_{gp}(x_3) = 1$ for all $p \in A$. Also

$$\mu_{f_p+g_p}(x_1 + x_3) \geq \sigma(\mu_{f_p}(x_1), \mu_{g_p}(x_3)) \geq 1,$$

hence, $\mu_{f_p+g_p}(x_1 + x_3) = 1$ and $f + g$ is certainly bounded.

Now let $\alpha \in \Re$ and $f \in L_c(V, V')$. Because of (PN2), it suffices to consider the case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αf is certainly bounded.

Proceeding by induction, assume that $\alpha f \in L_c(V, R)$ for $\alpha = 0, 1, 2, \ldots, n-1$ with $n \in \mathbb{N}$. Then for every $p \in A$,

$$\mu_{nf_p} \geq \sigma(\mu_{(n-1)f_p}, \mu_{f_p}) \geq 1,$$

so that $\mu_{nf_p} = 1$, i.e. αf is certainly bounded. Therefore αf is certainly bounded for every positive integer n. If α is not a positive integer, there is $n \in \mathbb{Z}$, such that $n-1 < \alpha < n$; therefore by Lemma 1.1, every $p \in A$ one has

$$\mu_{\alpha fp} \geq \mu_{nf_p},$$

and whence

$$\mu_{\alpha fp} = 1,$$

so that αf is certainly bounded. □

Theorem 3.2. *$L_D(V, V')$ is a linear space, where $\sigma(D^+, D^+) \subset D^+$.*

Proof. Let f and g be two D-bounded linear operators from (V, v, τ, τ^*) into $(V', \mu, \sigma, \sigma^*)$. Thus, R'_{fA} and R'_{gA} are in D^+. Since, for every $p \in A$, one has

$$\mu_{fp+gp} \geq \sigma(\mu_{f_p}, \mu_{g_p}) \geq \sigma(R'_{fA}, R'_{gA}),$$

which belong to D^+, also $R'_{(f+g)A}$ belong to D^+ and $f+g$ is D-bounded. Now let $\alpha \in \Re$ and $f \in L_D(V, V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αf is strongly D-bounded.

Proceeding by induction, assume that $\alpha f \in L_D(V, V')$, i.e. $R'_{\alpha fA} \in D^+$ for $\alpha = 0, 1, 2, \ldots, n-1$ with $n \in \mathbb{N}$. Then, for every $p \in A$,

$$\mu_{nfp} \geq \sigma(\mu_{(n-1)fp}, \mu_{fp}),$$

and hence

$$R'_{nfp} \geq \sigma(R'_{(n-1)fA}, R'_{fA}),$$

$R'_{nfA} \in D^+$ and nf is D-bounded. Therefore nf is D-bounded for every positive integer n. If α is not a positive integer, there is $n \in \mathbb{Z}$, such that $n-1 < \alpha < n$; therefore Lemma 1.1 , every $p \in A$ one has $\mu_{\alpha f_p} \geq \mu_{nf_p}$, whence $R_{\alpha A} \geq R_{nfA}$ which means that αf is D-bounded. □

Theorem 3.3. *When $\sigma = \min$, then $L_B(V, V')$ is a linear space.*

Proof. Let f and g be two strongly B-bounded linear operators from (V, v, τ, τ^*) into $(V', \mu, \sigma, \sigma^*)$. By Definition 2.4, for every $p \in V$ and $x > 0$, there exist $k_1, k_1 > 0$ such that:

$$\mu_{fp}(x) \geq v_p(\frac{x}{k_1}) \ , \tag{3.1}$$

and

$$\mu_{gp}(x) \geq v_p(\frac{x}{k_2}) \ . \tag{3.2}$$

From (3.1) and (3.2) we note that:

$$\mu_{(f+g)p}(x) = \mu_{fp+gp}(x) \geq \sigma(\mu_{fp}(\tfrac{x}{2}), \mu_{gp}(\tfrac{x}{2})) \ \geq \sigma((v_{fp}(\tfrac{x}{2k_1}), v_{gp}(\tfrac{x}{2k_2})) \ . \tag{3.3}$$

Choose $k = \max\{2k_1, 2k_2\} + 1$. Thus, $k \geq 2k_1$ and $k \geq 2k_2$, this implies that $\frac{x}{2k_1} \geq \frac{x}{k}$ and $\frac{x}{2k_2} \geq \frac{x}{k}$, $\forall\, x \geq 0$. Thus

$$v_p(\tfrac{x}{2k_1}) \geq v_p(\tfrac{x}{k}) \text{ and } v_p(\tfrac{x}{2k_2}) \geq v_p(\tfrac{x}{k}),\ \forall x \geq 0.$$

Thus $\min\left\{v_p(\frac{x}{2k_1}), v_p(\frac{x}{2k_2})\right\} \geq v_p(\frac{x}{k})$.

Now from (3.3) we get $\mu_{(f+g)p}(x) \geq v_p(\frac{x}{k})$, $\forall x \geq 0$, so that $f+g$ is strongly B-bounded.

Now let $\alpha \in \Re$ and $f \in L_B(V, V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αf is strongly B-bounded.

Proceeding by induction, assume that $\alpha f \in L_B(V, V')$ for $\alpha = 0, 1, 2, 3, .., n-1$ with $n(\geq 2) \in N$ from (3.1) and (3.2), for every $p \in A$, then

$$\begin{aligned}&\mu_{nfp}(x) \geq \sigma(\mu_{(n-1)fp}(\tfrac{x}{2}), \mu_{fp}(\tfrac{x}{2})) \geq \sigma(v_{(n-1)p}(\tfrac{x}{2k_1}), v_p(\tfrac{x}{2k_2}))\\ &\quad \geq \sigma(v_{(n-1)p}(\tfrac{x}{k}), v_p(\tfrac{x}{k})) \ \geq \min(v_{(n-1)p}(\tfrac{x}{k}), v_p(\tfrac{x}{k})) = v_{(n-1)p}(\tfrac{x}{k}) > v_{np}(\tfrac{x}{k}).\end{aligned}$$

So that αf is strongly B-bounded. Therefore αf is strongly B-bounded for every positive integer n. If α is not a positive integer, there is $n \in \mathbb{Z}$, such that $n-1 < \alpha < n$; therefore Lemma 1.1, every $p \in A$ one has $\mu_{\alpha f_p} \geq \mu_{nf_p}$, which means that αf is strongly B-bounded. This implies that $f+g$ is strongly B-bounded. Hence $L_B(V, V')$ is a linear space. □

Theorem 3.4. *When $\sigma = \min$, $L_\psi(V, V')$ is a linear space.*

Proof. Let f and g be two strongly ψ-bounded linear operator from (V, v, τ, τ^*) into $(V', \mu, \sigma, \sigma^*)$. By Definition 2.4, for every $p \in V$ and $x > 0$, there exist $\psi_1(x) < x$ and $\psi_2(y) < y$, such that

$$v_p(x) > 1 - x \Rightarrow \mu_{f_p}(\psi_1(x)) > 1 - \psi_1(x)\ ,$$

$$v_p(y) > 1 - y \Rightarrow \mu_{g_p}(\psi_2(y)) > 1 - \psi_2(y)\ .$$

Let $\psi(x+y) = \psi_1(x) + \psi_2(y) < x + y$. If $v_p(x+y) > 1 - (x+y)$, then

$$\begin{aligned}&\mu_{(f+g)_p}(\psi(x+y)) = \mu_{f_p+g_p}(\psi_1(x) + \psi_2(y))\\ &\quad \geq \sigma(\mu_{f_p}(\psi_1(x), \mu_{g_p}(\psi_2(y)))\\ &\quad \geq \min(\mu_{f_p}(\psi_1(x), \mu_{g_p}(\psi_2(y)))\\ &\quad > 1 - \psi(x+y).\end{aligned}$$

So $f+g$ is strongly ψ-bounded.

Now let $\alpha \in \Re$ and $f \in L_\psi(V, V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αf is strongly ψ-bounded.

Proceeding by induction, assume that $\alpha f \in L_\psi(V, V')$ for $\alpha = 0, 1, 2, \ldots, n-1$ with $n \in$ N. Then, for every $p \in A$, let $\psi(x+y) > \psi_1(x) + \psi_2(y)$

$$\mu_{nf_p}(\psi(x+y)) \geq \sigma(\mu_{(n-1)f_p}(\Psi_1(x), \Psi_2(y)) > 1 - \Psi(x+y).$$

So that αf is strongly ψ-bounded. Therefore αf is strongly ψ-bounded for every positive integer n. If α is not a positive integer, there is $n \in \mathbb{Z}$, such that $n-1 < \alpha < n$; therefore Lemma 1.1, every $p \in A$ one has $\mu_{\alpha f_p} \geq \mu_{ng_p}$, which means that αf is strongly ψ-bounded. □

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