Holomorphic Distribution of CR-Submanifolds of a Complex Projective Space

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Abstract. In this paper, we characterize CR-submanifolds of a complex projective space with parallel mean curvature vector and flat normal connection under certain symmetry conditions on the shape operator in the holomorphic distribution.

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1. Introduction

Let $N$ be a Kaehler manifold with complex structure $J$ and Riemannian metric $g$. A submanifold $M$ of $N$ is called a CR-submanifold if there exists on $M$ a holomorphic distribution $D$ such that its complementary orthogonal distribution $D^\perp$ is anti-invariant by $J$, i.e., $JD_x = D_x$ and $JD_x^\perp \subseteq T_x M^\perp$, for $x \in M$ (cf. [2]). It follows that the normal bundle splits as $TM^\perp = JD^\perp \oplus \nu$, where $\nu$ is an invariant subbundle of $TM^\perp$ by $J$. If $D = \{0\}$ (resp. $D^\perp = \{0\}$) then $M$ is said to be an anti-invariant (resp. invariant) submanifold.

In what follows, we denote by $\Gamma(V)$ the module of all differentiable sections on a vector bundle $V$ over $M$.

For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, we denote by $PX$ and $FX$ the tangential part and the normal part of $JX$ respectively, while $t\xi$ and $f\xi$ are the tangential part and the normal part of $J\xi$ respectively.

An important class of CR-submanifolds is the class of all real hypersurfaces of a Kaehler manifold. In particular, real hypersurfaces of a complex space form has been studied extensively (for details see [7]). In [6], Ki and Suh obtained a characterization on real hypersurfaces $M$ of a complex space form $M_n(c)$ with constant holomorphic sectional curvature $4c \neq 0$ under the following conditions:

$$(\nabla_X A)Y = cg(PX, Y)JC,$$

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\[(AP - PA)X = -\tau(X)JC\]

for any \(X, Y \in \Gamma(D)\), where \(C\) is a unit normal vector field of \(M\), \(A\) is the shape operator of \(M\) and \(\tau\) is a 1-form on \(M\). The purpose of this paper is to extend Ki and Suh’s result to the setting of CR-submanifolds of a complex projective space \(CP^n\), i.e., we have

**Theorem 1.1.** Let \(M\) be an \(m\)-dimensional complete, connected CR-submanifold of \(CP^n\) with flat normal connection and parallel mean curvature vector. Suppose that the isometric immersion of \(M\) into \(CP^n\) is full and \(M\) satisfies the following two conditions:

1. \((\nabla_X A)\xi Y = cg(PX, Y)t \xi,\)
2. \((A_{\xi} P - PA_{\xi})X = -\tau(X)\xi\)

for any \(X, Y \in \Gamma(D)\) and \(\xi \in \Gamma(TM^\perp)\), where \(\tau\) is a 1-form on \(M\). Then \(M\) is either

(a) \(\pi(S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{m+1}))\) of \(CP^n\), where \(\sum_{i=1}^{m+1} r_i^2 = 1\) and \(n = m\) or

(b) \(\pi(S^{n_1}(r_1) \times S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_k))\) of \(CP^n\), where \(\sum_{i=1}^{k} r_i^2 = 1\) and \(\sum_{i=1}^{k} n_i = m+1\). Here \(n_1, n_2, \ldots, n_k\) are odd numbers except for the case of \(n_1 = n_2 = \cdots = n_k = 1\), and \(2n + 1 = m + k\).

**Remark.** The submanifold (a) in Theorem 1.1 has parallel second fundamental form \(h\). On the other hand, the second fundamental form \(h\) of the submanifold (b) is not parallel. However, it is cyclic parallel, namely this second fundamental form \(h\) satisfies

\[g(\nabla_X h)Y, Z) + g(\nabla_Y h)Z, X) + g(\nabla_Z h)X, Y) = 0\]

for all vectors \(X, Y\) and \(Z\) on the submanifold (b). It is well-known that this equation is equivalent to the following:

\[g(\nabla_X h)X, X) = 0\]

for each vector \(X\) on the submanifold (b). The spaces in the above theorem are described as follows:

Let \(S^n(r)\) be an \(n\)-dimensional sphere with radius \(r\) center at the origin and \(CP^n\) the complex projective space of complex dimension \(n\) with constant holomorphic sectional curvature curvature 4c. Consider the following commutative diagram (cf. [9, page 223]):

\[
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\iota} & S^{2n+1} \\
\pi \downarrow & & \bar{\pi} \downarrow \\
M & \xrightarrow{\iota} & CP^n \\
\end{array}
\]

where \(\bar{\pi}\) is the Hopf fibration from the unit sphere \(S^{2n+1}\) onto \(CP^n\). If \(\widetilde{M}\) is a contact CR-submanifold in \(S^{2n+1}\) then the submersion \(\pi = \pi|_{\widetilde{M}}\) induced a CR-submanifold \(M = \pi(\widetilde{M})\) on \(CP^n\). In particular, when we put \(\widetilde{M} = S^{n_1}(r_1) \times \cdots \times S^{n_k}(r_k)\) of \(CP^n\), then \(M = \pi(\widetilde{M})\) is a.
$S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_{m+1})$, where $\sum_{i=1}^{k} r_i = 1$, $n_1, n_2, \ldots, n_k$ are odd numbers, $\sum_{i=1}^{k} n_i = m + 1$ and $m + k = 2n + 1$. Then $M = \pi(\tilde{M})$ is a CR-submanifold of $CP^n$ with parallel mean curvature vector and with flat normal connection.

2. Preliminaries

Let $N$ be a Kaehler manifold with complex structure $J$ and with Riemannian metric $g$. Suppose $M$ is an $m$-dimensional Riemannian manifold isometrically immersed in $N$. We denote by the same $g$ the Riemannian metric induced on $M$, $\nabla$ and $\tilde{\nabla}$ respectively the Levi-Civita connection on $N$ and the connection induced on $M$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_XY = \nabla_XY + h(X,Y)$$

$$\tilde{\nabla}_X\xi = -A_\xi X + \nabla^\perp_X\xi$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, where $\nabla^\perp$ denotes the normal connection induced in the normal bundle $TM^\perp$ of $M$ and $h$ the second fundamental form of $M$. The shape operator $A_\xi$ is related to $h$ by

$$g(A_\xi X, Y) = g(h(X,Y), \xi).$$

We define the covariant derivative of $A$ by

$$\nabla_X(A_\xi Y) = \nabla_X(A_\xi Y) - A_\nabla^\perp_\xi Y - A_\xi \nabla_X Y.$$

Then we have

$$g((\nabla_X A)\xi Y, Z) = g((\nabla_X h)(Y, Z), \xi)$$

for any $X, Y, Z \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$. Let $R^\perp$ be the curvature tensor associated with $\nabla^\perp$. If $R^\perp = 0$ then we say that the normal connection $\nabla^\perp$ is flat. A normal vector field $\xi$ is said to be parallel if we have $\nabla^\perp\xi = 0$. Let $M$ be a CR-submanifold of a Kaehler manifold $N$. Then we have the following identities [3]

(2.1) $$(\nabla_X P)Y = A_{FY}X + th(X,Y)$$

(2.2) $$(\nabla_X F)Y = -h(X, PY) + fh(X,Y)$$

(2.3) $$(\nabla_X t)\xi = A_{f\xi}X - PA_\xi X$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$.

When the ambient space $N$ is the complex projective space $CP^n$ of constant holomorphic sectional curvature $4c > 0$, the Codazzi equation is given as follows

(2.4) $$(\nabla_X A)\xi Y - (\nabla_Y A)\xi X = c\{g(FX, \xi)PY - g(FY, \xi)PX + 2g(PX, Y)t\xi\}.$$
Theorem 2.1. \[8\] Let \(M\) be an \(m\)-dimensional complete, connected CR-submanifold of \(CP^n\) with flat normal connection and parallel mean curvature vector. If \(A_\xi P = PA_\xi\), for any \(\xi \in \Gamma(TM^\perp)\), then \(M\) is either
\[(a) \quad \pi(S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{m+1})); \quad \sum_{i=1}^{m+1} r_i^2 = 1 \] of some \(CP^m\) in \(CP^n\); or
\[(b) \quad \pi(S^{n_1}(r_1) \times S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_k)); \quad \sum_{i=1}^{k} r_i^2 = 1 ; \quad \sum_{i=1}^{k} n_i = m + 1 \text{ where } n_1, n_2, \cdots, n_k \text{ are odd numbers and } 2n + 1 = m + k.\]

Lemma 2.1. \[1\] Let \(M\) be an \(m\)-dimensional complete, connected, totally geodesic submanifold of \(CP^n\). Then \(M\) is either
\[(a) \text{ an invariant complex projective space } CP^{m/2}; \text{ or; }\]
\[(b) \text{ an anti-invariant real projective space } RP^m.\]

Theorem 2.2. \[5\] Let \(M\) be an invariant submanifold of a complex space form \(M_n(c)\). Then the normal connection is flat if and only if \(c = 0\) and \(M\) is totally geodesic.

3. Certain lemmas

In order to prove our Theorem, we prepare some lemmas in this section. We first prove the following:

Lemma 3.1. Let \(M\) be a CR-submanifold of a Kaehler manifold \(N\). If \(M\) satisfies the condition \((1.2)\) then \(A_\xi X = 0\), for any \(X \in \Gamma(D)\) and \(\xi \in \Gamma(\nu)\).

Proof. By putting \(X \in \Gamma(D)\), \(\xi \in \Gamma(\nu)\) in \((1.2)\) and taking inner product with \(Y \in \Gamma(TM)\), we obtain
\[
(3.1) \quad g((A_\xi P - PA_\xi)X, Y) = -\tau(X)g(t_\xi, Y) = 0
\]
since \(t_\xi = 0\). If we let \(Y \in \Gamma(D^\perp)\) then we have \(g(A_\xi PX, Y) = 0\), which means that \(A_\xi D \perp D^\perp\). Next, by putting \(X, Y \in \Gamma(D)\) in \((2.2)\) and taking inner product with \(\xi \in \Gamma(\nu)\) we have
\[
0 = g(-h(X, PY) + fh(X, Y), \xi) = -g(A_\xi X, PY) - g(A_f_\xi X, Y).
\]
Since \(A_f_\xi\) is symmetric with respect to \(X\) and \(Y\), the above equation and \((3.1)\) imply that \(g(A_\xi PX, Y) = 0\) for any \(X, Y \in \Gamma(D)\) and \(\xi \in \Gamma(\nu)\). Hence \(A_\xi D \perp D\) and this completes the proof. \(\square\)

Now let \(M\) be an \(m\)-dimensional CR-submanifold of \(CP^n\) with flat normal connection. Then there is a local field of orthonormal frames \(\{\xi_a\}\) in \(\Gamma(TM^\perp)\) such that each \(\xi_a, (1 \leq a \leq 2n - m)\) is parallel, (cf. \([4, \text{ page } 99]\)). In the following we put \(A_a = A_{\xi_a}\) and
\[
(\nabla_X A)_a = (\nabla_X A)_{\xi_a}
\]
to simplify the notation.

Lemma 3.2. Let \(M\) be a CR-submanifold of \(CP^n\) with flat normal connection. If \(M\) satisfies the conditions \((1.1)\) and \((1.2)\) then
\[
\tau(Y)g(A_a X, PZ) + \tau(PY)g(A_a X, Z) + \tau(Z)g(A_a X, PY) + \tau(PZ)g(A_a X, Y) = 0
\]
for any \(X, Y, Z \in \Gamma(D)\).
Proof. For any \( Y, Z \in \Gamma(D) \), (1.2) implies that 
\[ g((A_a P - P A_a) Y, Z) = 0. \]
Differentiating this equation with respect to \( X \in \Gamma(D) \), we get
\[
g((\nabla_X A)_a Z, PY) + g((\nabla_X A)_a Y, PZ) + g((\nabla_X P)Z, A_a Y) + g((\nabla_X P)Y, A_a Z)
\]
(3.2)
\[ + g(\nabla_X Z, (A_a P - P A_a) Y) + g(\nabla_X Y, (A_a P - P A_a) Z) = 0. \]
On the other hand, by using (1.2), we have
\[ g(\nabla_X Z, (A_a P - P A_a) Y) = -\tau(Y) g(\nabla_X Z, t\xi_a) = \tau(Y) g(Z, (\nabla_X t)\xi_a). \]
Together with (2.3) and Lemma 3.1, yields
\[ g(\nabla_X Z, (A_a P - P A_a) Y) = \tau(Y) g(Z, A_{f_a}X - P A_a X) = \tau(Y) g(P Z, A_a X). \]
Similarly, we have
\[ g(\nabla_X Y, (A_a P - P A_a) Z) = \tau(Z) g(P Y, A_a X). \]
It follows from (1.1), (2.1) and these equations that (3.2) reduces to
\[
g(th(X, Z), A_a Y) + g(th(X, Y), A_a Z)
\]
(3.3)
\[ + \tau(Y) g(A_a X, P Z) + \tau(Z) g(A_a X, P Y) = 0. \]
Note that the condition (1.2) implies that \( FA_a X = \tau(P X) F t\xi_a \). From which, together with Lemma 3.1 and the fact that
\[ \xi_a = -F t\xi_a - f^2 \xi_a \]
we get
\[ g(th(X, Z), A_a Y) = \tau(P Y) g(A_a X, Z), \]
and
\[ g(th(X, Y), A_a Z) = \tau(P Z) g(A_a X, Y). \]
From these equations and (3.3), we obtain the lemma. \( \square \)

4. Proof of the theorem

First of all, let us suppose that \( t\xi_a = 0 \) for all \( a \). Then \( M \) is an invariant submanifold of \( CP^n \), but this contradicts Theorem 2.2. Hence we may assume that \( t\xi_a \neq 0 \) for some \( a \) and define a local vector field \( V_a = -\frac{1}{\|t\xi_a\|} t\xi_a \). Now we decompose \( A_a V_a \) into
\[
A_a V_a = \beta_a U_a + \mu_a W_a
\]
where \( U_a \) and \( W_a \) are two unit vector fields in \( \Gamma(D) \) and \( \Gamma(D^\perp) \) respectively.

Next we shall show that \( \beta_a = 0 \). For this purpose, we suppose that there exists a non empty open set \( G \) in \( M \) on which \( \beta_a \) is nowhere zero. Then taking inner product with \( t\xi_a \) in (1.2) and using the above equation, we obtain
\[
\tau(X) = -\beta_a g(X, P U_a)
\]
(4.2)
for any \( X \in \Gamma(D) \). From which we can see that \( \tau(P U_a) = -\beta_a \neq 0 \) and \( \tau(U_a) = 0 \). By putting \( Z = U_a \) in Lemma 3.2, we obtain
\[ \tau(Y) g(A_a X, P U_a) + \tau(P Y) g(A_a X, U_a) + \tau(P U_a) g(A_a X, Y) = 0. \]
By putting \( Y = U \) and then \( Y = PU \) in this equation we obtain

\[
g(A_a PU_a, X) = g(A_a U, X) = 0
\]

for any \( X \in \Gamma(D) \), or equivalently, \( PA_a PU_a = PA_a U_a = 0 \). Together with (1.2) and (4.2) yields

\[
A_a PU_a = 0
\]

and

\[
(4.3) \quad A_a U_a = -\beta_a t \xi_a.
\]

Now by using the Codazzi equation, we get

\[
g((\nabla_{V_a} A)_a U_a, PU_a) - g((\nabla_{U_a} A)_a V_a, PU_a) = cg(FV_a, \xi_a) = c.
\]

On the other hand, (4.3) implies that

\[
g((\nabla_{V_a} A)_a U_a, PU_a) - g((\nabla_{U_a} A)_a V_a, PU_a) = g(A_a \nabla_{V_a} U_a, PU_a) = -\beta_a g((\nabla_{V_a} t) \xi_a, PU_a).
\]

By using (2.3), (4.1) and Lemma 3.1, this equation gives

\[
g((\nabla_{V_a} A)_a U_a, PU_a) = \beta_a^2.
\]

Moreover, it follows from (1.1) that we can see

\[
g((\nabla_{U_a} A)_a V_a, PU_a) = g((\nabla_{U_a} A)_a PU_a, V_a) = cg(t \xi_a, V_a) = -c.
\]

From these equations, we obtain \( \beta_a^2 = 0 \), which is a contradiction. Consequently, we conclude that \( \beta_a = 0 \) and so (4.2) implies that \( \tau(X) = 0 \) for \( X \in \Gamma(D) \). Since \( (A_a P - PA_a) D^\perp = 0 \), we have \( A_a P - PA_a = 0 \). Thus our Theorem follows from Theorem 2.1.

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References