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Compatible Factorizations and Three-fold Triple Systems

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Abstract. A three-fold triple system is a design wherein each pair of treatments occurs exactly once. One way to construct this design is by using an idempotent commutative quasigroup. This paper attempts to provide another method of constructing a 3-fold triple system. Firstly, we would like to discuss compatible factorization without multiple edges using a patterned starter construction. Then, we will use this construction to enumerate a distinct 3-fold triple system for every odd order $v > 3$.

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1. Introduction

The standard ideas and definitions of a graph are assumed. The complete graph on n vertices by k_n. A *one-factor* in a graph G refers to a set of edges in which every vertex appears exactly once, and the partition of the edge-set of G into *one-factors* is called a *one-factorization*. From this explanation it is obvious that for it to possess one-factorization, a graph with *one-factor* must have an even number of vertices.

If G has an odd number of vertices, then a *near-one-factor* consisting of one vertex (the focus) and a set of disjoint edges that contain every other vertex occurs. In such a situation, a *near-one-factorization* refers to a set of edge-disjoint near one factor that together contain all the edges.

Given the near-one-factor $N = x \quad ab \quad cd \quad \cdots \quad yz$ it will be convenient to refer it to $\{xab\}\{xcd\}\{xyz\}$ as the set of triples associated with N. A comprehensive discussion concerning one-factorization can be found in [2], [3], [7], [8], [9] and [10].

Wallis [11] introduces certain designs whose blocks are ordered triples, or triads subject to certain conditions. One of these designs, called design B, has seven treatments whose blocks are 35 triples of the treatment (refer to Table 1).

The blocks are arranged in seven rows in the following manner:

i) each block occurs exactly once in the design;
ii) each treatment occurs either twice or three times in each row;

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Table 1. A design for seven treatments

126 316 451	532	674
231 427 562	643	715
342 531 673	754	126
453 642 714	165	237
564 753 125	276	341
675 164 236	317	452
712 275 347	321	563

iii) no two treatments occur together in two or more blocks in any row;
iv) the blocks are ordered so that no treatment occur twice in the same position in any row.

If we consider column one in Table 1, we see each treatment occurs as a focus exactly once and each associated triples contains no repetition. In order to discuss this type of construction, we define such a set of near-one-factor to be a *compatible factorization* or CF. They form a "factorization" of the multigraph, with treatments as vertices, formed by taking the union of the factors with multiplicities preserved. This multigraph is called the graph of compatible factorization and is denoted as CF.

Now, it is natural to ask when compatible factorization exist. In the following sections, we formally define CF and examine the construction of CF. Then we would like to use this construction to construct a *3-fold triple system* such that each pair of treatments occurs exactly three times.

2. Compatible factorizations

Consider the following combinatorial problem: we are given n vertices and we wish to construct an array from a certain set of triples of these vertices; each triple must occur, at most, once in the array. We must arrange the elements in rows and columns such that in each row no pair of vertices is repeated and the number of rows equals the number of given vertices. For example, if $v = 7$ and we want to produce the following triples:

$$\{123\}, \{124\}, \{136\}, \{134\}, \{125\}, \{126\}, \{137\}, \{145\}, \{167\}, \{236\},$$
$$\{235\}, \{245\}, \{345\}, \{367\}, \{267\}, \{257\}, \{347\}, \{467\}, \{567\}, \{456\}, \{457\}.$$

We would have 21 different triples with seven vertices. Each vertex would appear precisely nine times but each pair of elements would not appear a constant number of times. Since we have seven elements, we shall have seven rows and consequently three columns. One possible construction is the following:

F1:	1	23	45	67
F2:	2	14	26	57
F3:	3	16	25	47
F4:	4	13	25	67
F5:	5	12	34	67
F6:	6	12	37	45
F7:	7	13	26	45
	C1	C2	C3	C4

In this case we have 7 rows and 4 columns. Now in order to form a triple, we append C1 with C2, C1 with C3, and C1 with C4 thus obtaining three triples in each row. For instance, in F1 we have triples $\{123\}$, $\{145\}$, $\{167\}$ and by continuing in the same fashion in F2 through F7, we will generate the desired triples. Any solution to this problem will be called a compatible factorization of order v and will be denoted as *CF (v).*

Definition 2.1. *A compatible factorization of order* v *or* $CF(v)$, *is an* $\frac{v(v-1)}{2}$ *array that satisfies the following conditions:*

i) *The entries in row i form a near-one-factor with focus i.*
ii) *The triples associated with the rows contain no repetitions.*

Note that the triples are unordered. For example $\{456\}$, $\{546\}$ are considered the same triple. An obvious necessary condition for the existence of $CF(v)$ is that v must be odd.

Theorem 2.1. *There exists a compatible factorization for every odd order* v *>3.*

Proof. Suppose $v = 2t + 1$ >3. Therefore, the near-one-factor from the patterned starter, with i-th factor is given as:

$$i(i+1)(i-1)(i+2)(i-2)\cdots(i+n)(i-n) \mod v$$

and is a compatible factorization. □

No *CF*(3) can exist: with the three symbols 1,2,3 the only possible near-one-factor with focus 1 being 1 23, the only possible near-one-factor with focus 2 being 2 13, and these two have a common associated triple.

By Theorem 2.1, the first case is order five and there is a unique *CF*(5) up to isomorphism. Complete searches show there are 62,800 *CF*(7) of which only 231 can be classified up to isomorphism [6] with the number for *CF(v)* getting arbitrarily large for other orders.

3. Construction of compatible factorization

Construction of Example 1 is done by trial and error by iterating, in each row, a pair of vertices in union with an isolated vertex, forming a triple not used in the previous row. The two cases that will be considered in *CF* construction are: (i) no multiple edges, (ii) multiple edges.

However, this paper focuses on constructions with no multiple edges. A discussion on construction with multiple edges can be found in [5].

Definition 3.1. *A compatible factorization with no multiple edges is a near-one-factor for which the totality of associated triples contains no repetitions.*

Theorem 3.1. *There exist a compatible factorization with no multiple edges for every odd orders $v > 3$.*

Proof. This follows from the proof of the Theorem 2.1. □

Table 2 and 3 display examples with no multiple edges of *CF*(5) and *CF*(7) respectively. From Table 2, the associated triples are {125}, {134}, {231}, {245}, {342}, {351}, {453}, {412}, {514}, and {523}. Form Table 3, the associated triples are {127}, {136}, {145}, ..., {734}. For *CF*(7) with no multiple edges, there are exactly two *CF*(7) up to isomorphism [5].

Table 2. A unique *CF*(5)

1	25	34
2	31	45
3	42	51
4	53	12
5	14	23

Table 3. *CF*(7) with no multiple edges

1	27	36	45
2	31	47	56
3	42	51	67
4	53	62	71
5	64	73	12
6	75	14	23
7	16	25	34

4. Three-fold triple system

Definition 4.1. *A λ-fold triple system is a pair (S, T), where S is a finite set and T is a collection of 3-element subsets of S called triples such that each pair of distinct elements of S belongs to exactly λ triples of T.*

A triple system in which each pair appears exactly once is called a *Steiner Triple System* (STS). Such a system exists for all $v \equiv 1$ or $3 \pmod 6$. In 1961, Hanani [4] determined that a triple system of n elements and λ-fold exists if and only if $\lambda(n-1) \equiv 0 (\mathrm{mod}\ 6)$ and $\lambda n(n-1) \equiv 0 (\mathrm{mod}\ 6)$.

In this section we would like to construct a triple system such that each pair of treatments occurs exactly three times (the 3-fold triple system). Previous constructions of the 3-fold triple system utilised idempotent commutative quasigroups [7]. In contrast, we would like to provide another alternative method in the construction of a 3-fold triple system.

Table 4. Commutative idempotent quasigroup of order 7

$\circ$	1	2	3	4	5	6	7
1	1	6	4	2	7	5	3
2	4	2	7	5	3	1	6
3	7	5	3	1	6	4	2
4	3	1	6	4	2	7	5
5	6	4	2	7	5	3	1
6	2	7	5	3	1	6	4
7	5	3	1	6	4	2	7

Case 1. 3-fold triple system construction using idempotent commutative quasigroup

Definition 4.2. *A quasigroup of order n is a pair $(Q, \circ)$, where Q is a set of size n and "$\circ$" is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions.*

Definition 4.3. *A quasigroup is said to be idempotent if cell (i, i) contains symbol i for $1 \leq i \leq n$. A quasigroup is said to be commutative if cells (i, j) and (j, i) contain the same symbol, for all $1 \leq i,\ j \leq n$.*

Theorem 4.1. *There exist a 3-fold triple system of every odd order $v > 1$.*

Proof. The following construction provides a proof of this theorem. The 3-fold construction. Let $(Q,\ \circ\ )$ be an idempotent commutative quasigroup of order v. Let $T = \{\{a,\ b,\ a \circ\ b\}\ | a < b \in\ Q\}$. Then $(Q,\ T)$ is a 3-fold triple system of order v. □

By using Table 4 and Theorem 3, we have the following 3-fold triple system.

125	234	375
132	257	451
146	264	451
153	271	472
167	347	562
174	354	576
236	361	673

Case 2. 3-fold triple system construction using compatible factorization

Theorem 4.2. *There exist a 3-fold triple system of every odd order $v > 3$ with a distinct triple.*

Proof. Suppose $v = 2t + 1 > 3$. Let Q be a compatible factorization from near-one-factor from the patterned starter with i-th factor by Theorem 2.1. Now let $T = \{i(i+1)(i-1), i(i+2)(i-2), \cdots i(i+n)(i-n)\} \mod v$. Then (Q, T) is a 3-fold triple system with a distinct triple. □

In this case, each triple occurs exactly once and each pair occurs exactly three times.

Table 5. Compatible factorization of order 7 and the associated 3-fold triple systems

1	27	36	45
2	31	47	56
3	42	51	67
4	53	62	71
5	64	73	12
6	75	14	23
7	16	25	34

127	136	145
231	247	256
342	351	367
453	462	471
564	573	512
675	614	623
716	725	734

5. Concluding remarks

We have constructed compatible factorization without multiple edges from the generalization of near-one-factorization. We have also proven the existence for such factorization. Then we discussed two different approaches in designing a 3-fold triple system. The first case used a commutative idempotent quasigroup approach while the alternative method used compatible factorization approach. The difference between the two being that the alternative method provides for a distinct triple (refer Definition 2.1) which explains for the non-existence of the case 3-fold triple system for order three when compatible factorizations are used.

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