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Direct Gradient Descent Control as a Dynamic Feedback Control for Linear System

J. NAIBORHU, S.M. NABABAN, R. SARAGIH AND I. PRANOTO
Department of Mathematics, Institut Teknologi Bandung
Jl. Ganesa 10, Bandung 40132, Indonesia

Abstract. In this paper we study about stabilization of linear control system by using the dynamic state/output feedback control. We can prove that the dynamic state feedback control exists if the static state feedback control exists. But the existence of the dynamic output feedback control can not be guaranteed only by the existence of static feedback control, passive condition is needed. By applying the direct gradient descent control as a dynamic state feedback control to linear system, we get a condition for guaranteing the asymptotic stability of the system.

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1. Introduction

A famous method in linear system to obtain arbitrary asymptotic behavior is Pole-Assignment method. In this paper we consider the stabilization problem of linear system by using the dynamic feedback control. The definition of dynamic feedback control is to add an integrator into the system. This definition is different with the one proposed by Isidori [3] or Sontag [10].

Consider the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u},\ \boldsymbol{x} \in \mathcal{R}^n,\ \boldsymbol{u} \in \mathcal{R}^r, \tag{1.1}$$

$$\boldsymbol{y} = C\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{R}^m, \tag{1.2}$$

where A, B and C are real matrices of dimension $(n \times n), (n \times m)$ and $(p \times n)$.

We are interested in finding a dynamic feedback control law

$$\dot{\boldsymbol{u}} = K_1\boldsymbol{x} + K_2\boldsymbol{u} \tag{1.3}$$

or

$$\dot{\boldsymbol{u}} = L_1\dot{\boldsymbol{y}} + L_2\boldsymbol{y} + L_3\boldsymbol{u} \tag{1.4}$$

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that makes the extended system

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \dot{\boldsymbol{u}} &= K_1\boldsymbol{x} + K_2\boldsymbol{u} \end{aligned} \tag{1.5}$$

or

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \dot{\boldsymbol{u}} &= L_1\dot{\boldsymbol{y}} + L_2\boldsymbol{y} + L_3\boldsymbol{u}, \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{aligned} \tag{1.6}$$

is asymptotically stable about $(\boldsymbol{x}, \boldsymbol{u}) = (0, 0)$.

In [11], it has been proved that if nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{1.7}$$

is locally asymptotically stabilizable with a $\mathcal{C}^1$ feedback law $\boldsymbol{u}(\boldsymbol{x})$ then

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \tag{1.8}$$

$$\dot{\boldsymbol{u}} = \boldsymbol{v} \tag{1.9}$$

is locally asymptotically stabilizable. Coron and Praly [2] studied more in detail about the relationship of stabilizability between the system in equation (1.7) and the extended system (1.8)-(1.9). In case where system (1.8)-(1.9) is considered as a cascade system, Krstic et.al.[4] calculated $\boldsymbol{v}$ by using backstepping method.

The purposes of this paper is as follows. Based on result in [4, 2, 11], in linear system, we show that the dynamic state/output feedback control exists if static state/output feedback control exists such that the system is asymptotically stable. Then, we apply the direct gradient descent control [6, 7] as the dynamic feedback control to stabilize the linear system.

2. Existence of dynamic feedback

2.1. Dynamic state feedback control. Consider a linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \ \boldsymbol{x} \in \mathcal{R}^n, \ \boldsymbol{u} \in \mathcal{R}^r \tag{2.1}$$

where A and B are real matrices of dimension $(n \times n)$ and $(n \times r)$.

Assume that there exists K such that $A + BK$ is Hurwitz. Then, by a static feedback control law

$$\boldsymbol{u} = K\boldsymbol{x}, \tag{2.2}$$

system (2.1) is asymptotically stable. However, if we take the time derivative of the static feedback control (2.2) then the extended system which is formed together with (2.1), i.e.,

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \dot{\boldsymbol{u}} &= KA\boldsymbol{x} + KB\boldsymbol{u} \end{aligned} \tag{2.3}$$

is not always asymptotically stable. In the following Proposition, however, we show that there exists a dynamic feedback controller of the form

$$\dot{\boldsymbol{u}} = K_1\boldsymbol{x} + K_2\boldsymbol{u}$$

when the static feedback control (2.2) exists.

Proposition 2.1. *Consider the linear system (2.1) and assume that there exists K such that $A + BK$ is Hurwitz. Then there exist K_1 and K_2, such that the extended system*

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \dot{\boldsymbol{u}} &= K_1\boldsymbol{x} + K_2\boldsymbol{u} \end{aligned} \tag{2.4}$$

becomes asymptotically stable about $(\boldsymbol{x}, \boldsymbol{u}) = (0, 0)$.

Proof. Consider the augmented system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{2.5}$$

$$\dot{\boldsymbol{u}} = \boldsymbol{v}. \tag{2.6}$$

Let $A_K = A + BK$. Since A_K is Hurwitz (Assumption) then there exists $P > 0$ for a given Q, $Q > 0$ such that Lyapunov equation below is satisfied.

$$P(A_K) + (A_K)^T P = -Q. \tag{2.7}$$

Let $V(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x}$ be a Lyapunov function of system (2.5), and

$$\boldsymbol{z} = \boldsymbol{u} - K\boldsymbol{x}. \tag{2.8}$$

Substitute $\boldsymbol{z}$ into (2.5)-(2.6) to obtain

$$\dot{\boldsymbol{x}} = A_K\boldsymbol{x} + B\boldsymbol{z}, \tag{2.9}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{v} - K(A_K\boldsymbol{x} + B\boldsymbol{z}). \tag{2.10}$$

Let $V_a(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{x}^T P\boldsymbol{x} + \boldsymbol{z}^T\boldsymbol{z}$ be a Lyapunov function of system (2.9)-(2.10). Then we have

$$\begin{aligned} \dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) &= \dot{\boldsymbol{x}}^T P\boldsymbol{x} + \boldsymbol{x}^T P\dot{\boldsymbol{x}} + 2\boldsymbol{z}^T\dot{\boldsymbol{z}} \\ &= \boldsymbol{x}^T(A_K^T P + PA_K)\boldsymbol{x} + 2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - K(A_K\boldsymbol{x} + B\boldsymbol{z})). \end{aligned} \tag{2.11}$$

From (2.7) we have $\boldsymbol{x}^T(A_K^T P + PA_K)\boldsymbol{x} = -\boldsymbol{x}^T Q\boldsymbol{x} < 0$. Our objective is to find $\boldsymbol{v}$ such that $\dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) < 0$. It holds if

$$2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - K(A_K\boldsymbol{x} + B\boldsymbol{z})) \leq 0. \tag{2.12}$$

There are many ways to find $\boldsymbol{v}$ such that equation (2.12) is satisfied. One of them is to find $\boldsymbol{v}$ such that

$$2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - K(A_K\boldsymbol{x} + B\boldsymbol{z})) = -2\boldsymbol{z}^T R\boldsymbol{z}. \tag{2.13}$$

To satisfy (2.13), it is sufficient if

$$\boldsymbol{v} = K(A_K\boldsymbol{x} + B\boldsymbol{z}) - B^T P\boldsymbol{x} - R\boldsymbol{z}. \tag{2.14}$$

So,

$$\dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) = -\boldsymbol{x}^T Q\boldsymbol{x} - 2\boldsymbol{z}^T R\boldsymbol{z} < 0. \tag{2.15}$$

Substitute (2.14) into (2.10) to obtain

$$\dot{\boldsymbol{x}} = A_K\boldsymbol{x} + B\boldsymbol{z} \tag{2.16}$$

$$\dot{\boldsymbol{z}} = -B^T P\boldsymbol{x} - R\boldsymbol{z}. \tag{2.17}$$

Since (2.15) holds, system (2.16)-(2.17) is asymptotically stable.

Substitute (2.8) into (2.14) to obtain

$$\begin{aligned} \boldsymbol{v} &= K(A_K\boldsymbol{x} + B(\boldsymbol{u} - K\boldsymbol{x})) - B^T P\boldsymbol{x} - R(\boldsymbol{u} - K\boldsymbol{x}) \\ &= (KA - B^T P + RK)\boldsymbol{x} + (KB - R)\boldsymbol{u}. \end{aligned} \tag{2.18}$$

Substitute (2.18) into (2.6). Then the resulted system in the $(\boldsymbol{x}, \boldsymbol{u})$ coordinates is

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{2.19}$$

$$\dot{\boldsymbol{u}} = (KA - B^T P + RK)\boldsymbol{x} + (KB - R)\boldsymbol{u}. \tag{2.20}$$

Since system (2.16)-(2.17) is equivalent to system (2.19)-(2.20), system (2.4) is asymptotically stable with $K_1 = KA - B^T P + RK$ and $K_2 = KB - R$. □

In the proof of Proposition 2.1 we can see the stability improvement of the system (2.1). Consider equation (2.1) as an original system and equations (2.19)-(2.20) as an extended system. For the original system, the Lyapunov function is $V(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x}$ and for the extended system, the Lyapunov function is $V_a(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x} + (\boldsymbol{u} - K\boldsymbol{x})^T(\boldsymbol{u} - K\boldsymbol{x})$, where K is choosen such that $A + BK$ is Hurwitz. By using static feedback control $\boldsymbol{u} = K\boldsymbol{x}$, $\dot{V}(\boldsymbol{x}) = -\boldsymbol{x}^T Q\boldsymbol{x} < 0$, and based on this static feedback control, we get $\dot{\boldsymbol{u}} = (KA - B^T P + RK)\boldsymbol{x} + (KB - R)\boldsymbol{u}$, $\dot{V}_a(\boldsymbol{x}) = -\boldsymbol{x}^T Q\boldsymbol{x} - 2(\boldsymbol{u} - K\boldsymbol{x})^T R(\boldsymbol{u} - K\boldsymbol{x}) < 0$ and

$$\dot{V}_a(\boldsymbol{x}) \leq \dot{V}(\boldsymbol{x}) < 0. \tag{2.21}$$

It means that by adding an integrator into the system, the extended system goes to zero quicker than or at least same with the original system.

Example 2.1. Consider a linear system

$$\dot{x}(t) = x(t) + u(t).$$

By static control law

$$u_s(t) = -2x(t),$$

we have

$$x_s(t) = x_0 e^{-t},$$

and system becomes asymptotically stable.

By applying Proposition 2.1 to compute the dynamic feedback control, we have the extended system

$$\dot{x}(t) = x + u(t), \tag{2.22}$$

$$\dot{u}(t) = -4.5x(t) - 3u(t), \tag{2.23}$$

which has solution:

$$x_d(t) = x_0 e^{-t} \cos \frac{1}{2}\sqrt{2}t,$$

and

$$u_d(t) = -2x_0 e^{-t} \left(\cos \frac{1}{2}\sqrt{2}t + \frac{1}{4}\sqrt{2} \sin \frac{1}{2}\sqrt{2}t \right).$$

Based on this calculation we conclude that:

$$|x_d(t)| \leq |x_s(t)|.$$

In other words, the system with dynamic feedback control goes to zero quicker than the system with static control.

Example 2.2. Consider a linear system

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{2.24}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The unforced dynamics of the system (2.24) is unstable but the system (2.24) is controllable. Since system (2.24) is controllable, it can be stabilized by static feedback control. From optimal control problem with $V(\boldsymbol{x}(0), u(\cdot), 0) = \int_0^\infty (\boldsymbol{x}^T\boldsymbol{x} + u^2)dt$, we have

$$u(t) = K\boldsymbol{x}(t) = \begin{bmatrix} -0.412 & -4.4142 \end{bmatrix} \boldsymbol{x}(t). \tag{2.25}$$

Based on the static feedback control (2.25), we calculate the dynamic feedback control (2.20). First we find P such that $A_K^T P + PA_K = -Q$ where $Q = I$.

$$\begin{bmatrix} 0 & -1.412 \\ 1 & -2.4142 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1.412 & -2.4142 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2.824p_2 & p_1 - 2.4142p_2 - 1.412p_4 \\ p_1 - 2.4142p_2 - 1.412p_4 & 2p_2 - 4.8284p_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.26}$$

From equation (2.26) we have

$$P = \begin{bmatrix} \frac{3.8262}{2.824} & \frac{1}{2.824} \\ \frac{1}{2.824} & \frac{1}{2.824} \end{bmatrix}. \tag{2.27}$$

By taking $R = I$, we have

$$\begin{aligned} K_1 &= \begin{bmatrix} 4.4142 & -9.2404 \end{bmatrix} - \begin{bmatrix} \frac{1}{2.824} & \frac{1}{2.824} \end{bmatrix} + \begin{bmatrix} -0.412 & -4.4142 \end{bmatrix} \\ &= \begin{bmatrix} 4.0022 - \frac{1}{2.824} & -13.6546 - \frac{1}{2.824} \end{bmatrix}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} K_2 &= \begin{bmatrix} -0.412 & -4.4142 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1 \\ &= -5.4142. \end{aligned} \tag{2.29}$$

Then we have the extended system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{2.30}$$

$$\dot{u}(t) = (4.0022 - \frac{1}{2.824})x_1(t) + (-13.6546 - \frac{1}{2.824})x_2(t) - 5.4142u(t). \tag{2.31}$$

Simulation results are shown in Fig. 1 with initial condition: $x_1(0) = 1$; $x_2(0) = 0.5$; $u(0) = -2.6191$. From the simulation results we see that the extended system goes to zero quicker than its original system.

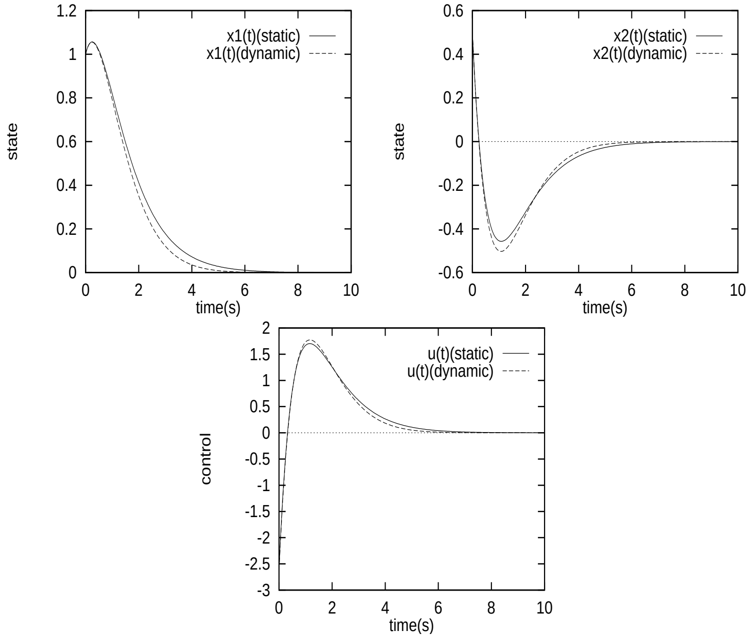

Figure 1. Static: static feedback control, dynamic: dynamic feedback control

2.2. Dynamic output feedback control. Consider the linear plant

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \ \boldsymbol{x} \in \mathcal{R}^n, \boldsymbol{u} \in \mathcal{R}^r, \tag{2.32}$$

$$\boldsymbol{y} = C\boldsymbol{x}, \ \boldsymbol{y} \in \mathcal{R}^m \tag{2.33}$$

where A, B and C are real matrices of dimension $(n \times n)$, $(n \times r)$ and $(m \times n)$.

It is well known that a stabilizing dynamic feedback compensator can be constructed if the system (2.32)-(2.33) is both stabilizable and detectable. For example, see [5]

In this section, we find K_1, K_2 and K_3 such that a dynamic controller of the form

$$\dot{\boldsymbol{u}} = K_1\dot{\boldsymbol{y}} + K_2\boldsymbol{y} + K_3\boldsymbol{u} \tag{2.34}$$

stabilizes the system (2.32) asymptotically.

Proposition 2.2. *Consider the linear plant (2.32)-(2.33) and assume that there exists K such that $A + BKC$ is Hurwitz and $B^TP = C$ with $P > 0$, $P = P^T$, and satisfy $P(A + BKC) + (A + BKC)^TP = -Q$ for a given $Q, Q > 0$. Then there exists K_1 , K_2 and K_3 such that the extended system*

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \dot{\boldsymbol{u}} &= K_1\dot{\boldsymbol{y}} + K_2\boldsymbol{y} + K_3\boldsymbol{u} \end{aligned} \tag{2.35}$$

becomes asymptotically stable about $(\boldsymbol{x}, \boldsymbol{u}) = (0, 0)$.

Proof. Consider the augmented system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \quad (2.36)$$

$$\dot{\boldsymbol{u}} = \boldsymbol{v}. \quad (2.37)$$

Let $\bar{K} = KC$ and $A_{\bar{K}} = A + B\bar{K}$. Since $A_{\bar{K}}$ is Hurwitz (Assumption) then there exists $P > 0$ for a given Q, $Q > 0$ such that the Lyapunov equation below is satisfied,

$$PA_{\bar{K}} + A_{\bar{K}}^T P = -Q. \quad (2.38)$$

Let $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T P\boldsymbol{x}$ be a Lyapunov function of system (2.36) and

$$\boldsymbol{z} = \boldsymbol{u} - \bar{K}\boldsymbol{x}. \quad (2.39)$$

Substitute $\boldsymbol{z}$ into (2.36)-(2.37) to obtain

$$\dot{\boldsymbol{x}} = A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z}, \quad (2.40)$$

$$\dot{\boldsymbol{z}} = \boldsymbol{v} - \bar{K}(A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z}). \quad (2.41)$$

Let $V_a(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{x}^T P\boldsymbol{x} + \boldsymbol{z}^T\boldsymbol{z}$ be a Lyapunov function of system (2.40)-(2.41). Then we have

$$\begin{aligned}\dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) &= \dot{\boldsymbol{x}}^T P\boldsymbol{x} + \boldsymbol{x}^T P\dot{\boldsymbol{x}} + 2\boldsymbol{z}^T\dot{\boldsymbol{z}} \\ &= \boldsymbol{x}^T(A_{\bar{K}}^T P + PA_{\bar{K}})\boldsymbol{x} + 2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - \bar{K}(A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z})). \quad (2.42)\end{aligned}$$

From (2.38) we have $\boldsymbol{x}^T(A_{\bar{K}}^T P + PA_{\bar{K}})\boldsymbol{x} = -\boldsymbol{x}^T Q\boldsymbol{x} < 0$. Our objective is to find $\boldsymbol{v}$ such that $\dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) < 0$. It holds if

$$2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - \bar{K}(A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z})) \leq 0. \quad (2.43)$$

There are many ways to find $\boldsymbol{v}$ such that equation (2.43) is satisfied. One of them is to find $\boldsymbol{v}$ such that

$$2\boldsymbol{z}^T B^T P\boldsymbol{x} + 2\boldsymbol{z}^T(\boldsymbol{v} - \bar{K}(A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z})) = -2\boldsymbol{z}^T R\boldsymbol{z}. \quad (2.44)$$

To satisfy (2.44), it is sufficient if

$$\boldsymbol{v} = \bar{K}(A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z}) - B^T P\boldsymbol{x} - R\boldsymbol{z}. \quad (2.45)$$

So,

$$\dot{V}_a(\boldsymbol{x}, \boldsymbol{z}) = -\boldsymbol{x}^T Q\boldsymbol{x} - 2\boldsymbol{z}^T R\boldsymbol{z} < 0. \quad (2.46)$$

Substitute (2.45) into (2.41) to obtain

$$\dot{\boldsymbol{x}} = A_{\bar{K}}\boldsymbol{x} + B\boldsymbol{z}, \quad (2.47)$$

$$\dot{\boldsymbol{z}} = -B^T P\boldsymbol{x} - R\boldsymbol{z}. \quad (2.48)$$

Since (2.46) holds, system (2.47)-(2.48) is asymptotically stable.

Substitute (2.39) into (2.45) to obtain

$$\begin{aligned}\boldsymbol{v} &= \bar{K}(A_{\bar{K}}\boldsymbol{x} + B(\boldsymbol{u} - \bar{K}\boldsymbol{x})) - B^T P\boldsymbol{x} - R(\boldsymbol{u} - \bar{K}\boldsymbol{x}) \\ &= (\bar{K}A - B^T P + R\bar{K})\boldsymbol{x} + (\bar{K}B - R)\boldsymbol{u}. \quad (2.49)\end{aligned}$$

By condition $B^T P = C$, (2.49) becomes

$$\begin{aligned}\boldsymbol{v} &= \bar{K}(A\boldsymbol{x} + B\boldsymbol{u}) + (-C + R\bar{K})\boldsymbol{x} - R\boldsymbol{u} \\ &= K\dot{\boldsymbol{y}} + (-I + RK)\boldsymbol{y} - R\boldsymbol{u}. \quad (2.50)\end{aligned}$$

Substitute (2.50) into (2.37). Then the resulted system in the $(\boldsymbol{x}, \boldsymbol{u})$ coordinates is

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{2.51}$$

$$\dot{\boldsymbol{u}} = K\dot{\boldsymbol{y}} + (-I + RK)\boldsymbol{y} - R\boldsymbol{u}. \tag{2.52}$$

Since system (2.47)-(2.48) is equivalent to system (2.51)-(2.52), system (2.35) is asymptotically stable with $K_1 = K$, $K_2 = -I + RK$ and $K_3 = -R$. □

From Proposition 2.2, the existence of static output feedback control is not enough to guarantee the existence of dynamic output feedback control (2.34). We need in proposition the condition $B^T P = C$ (passivity condition [8]) to guarantee the existence of dynamic output feedback control (2.34).

Example 2.3. Consider a linear plant

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{2.53}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{2.54}$$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}; \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The unforced dynamics of the system (2.53) is unstable but the system (2.53) is controllable. The linear plant (2.53)-(2.54) is unobservable. This system can be stabilized by static feedback output control

$$u(t) = Ky(t) = -2 \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t); \; \bar{K} = \begin{bmatrix} -2 & 0 \end{bmatrix}. \tag{2.55}$$

Based on static feedback output control (2.55), we stabilize the system (2.53) by a dynamic control

$$\dot{u}(t) = K_1 y(t) + K_2 u(t)$$

where

$$\begin{bmatrix} K_1 & 0 \end{bmatrix} = K_1 C = \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 \end{bmatrix},$$

and therefore

$$K_1 = -5. \tag{2.56}$$

Similarly

$$K_2 = \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 = -3. \tag{2.57}$$

Then we have

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \tag{2.58}$$

$$\dot{u}(t) = -5y(t) - 3u(t). \tag{2.59}$$

Simulation results are shown in Fig. 2 with initial condition: $x_1(0) = 1$; $x_2(0) = 0.5$; $u(0) = -2$. From simulation results we see that the extended system goes to zero quicker than its original system.

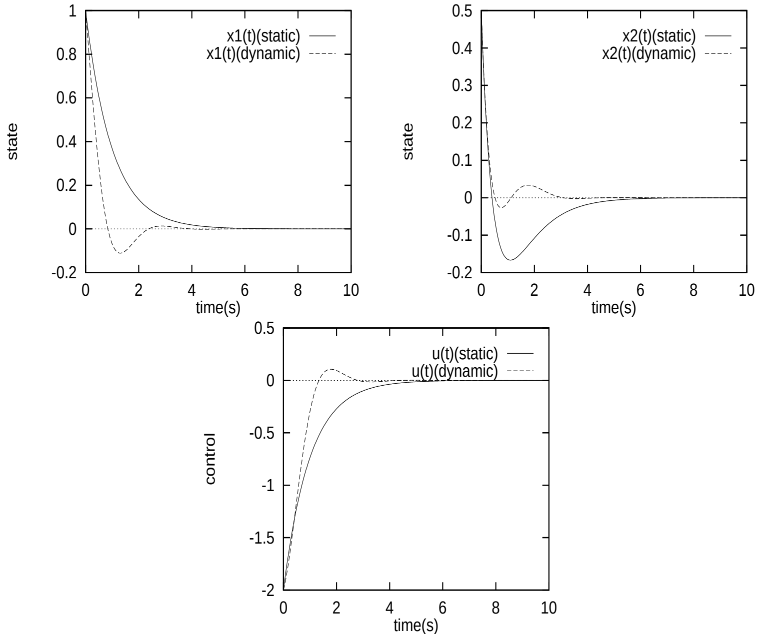

Figure 2. Static: $u(t) = K\boldsymbol{y}(t)$, dynamic: $\dot{u}(t) = K_1\boldsymbol{y}(t) + K_2u(t)$

3. Direct gradient descent control

Consider a nonlinear control system

$$\dot{\boldsymbol{x}}(t) = \mathbf{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)),\ \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3.1}$$

where $\boldsymbol{x}(t) \in \mathcal{R}^n$ is the state vector and $\boldsymbol{u}(t) \in \mathcal{R}^m$ is the control vector.

In general, the aim of control is to decrease a performance index $F(\boldsymbol{x}(t), \boldsymbol{u}(t))$ at any time t along the trajectory of system (3.1). In particular, our problem is formulated as follows:

$$\underset{\boldsymbol{u}(t)}{\text{decrease}} \qquad F(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{3.2}$$

$$\text{subj.to} \qquad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{3.3}$$

As the class of admissible controls, we consider a space $\mathcal{U}_t$ consisting of m-dimensional-vector valued functions which are continuous a.e. on $[t_0, t]$, and define the following inner product:

$$< \boldsymbol{u}, \boldsymbol{v} >= \int_{t_0}^{t} \boldsymbol{u}(\tau)^T \boldsymbol{v}(\tau) d\tau, \tag{3.4}$$

where $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathcal{U}_t$.

In simple form, the problem (3.2)-(3.3) can be written as

$$\underset{\boldsymbol{u}(t)}{\text{decrease}} \qquad F(\boldsymbol{x}(t; \boldsymbol{u}), \boldsymbol{u}(t)) \tag{3.5}$$

where

$$\boldsymbol{x}(t;\boldsymbol{u}) \quad = \quad \boldsymbol{x}(t_0)+\int_{t_0}^{t}\boldsymbol{f}(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))d\tau. \tag{3.6}$$

A very broad and perhaps the most important class of methods for unconstrained optimization is one that is based on the so-called gradient descent methods. By applying gradient descent method, we obtain as a control law, the first-order ordinary differential equation

$$\dot{\boldsymbol{u}}(t) = -\alpha\nabla_{\boldsymbol{u}}F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t)), \tag{3.7}$$

where α can become a constant or function of $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$ and t. Next, the gradient of performance index $F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))$ with respect to $\boldsymbol{u}(t)$ will be derived.

Assume that

A.1: $\boldsymbol{f}$ and F are continuously differentiable on $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{R}^n \times \mathcal{R}^m$.
A.2: $\boldsymbol{f}_{\boldsymbol{x}}$ and $\boldsymbol{f}_{\boldsymbol{u}}$ are Lipschitz continuous .

Definition 3.1. *Functional $\boldsymbol{x}(t;\boldsymbol{u})$ is Gateaux differentiable if for arbitrary $\boldsymbol{s} \in \mathcal{U}_t$,*

$$\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u}) = \frac{d}{d\varepsilon}\boldsymbol{x}(t:\boldsymbol{u}+\varepsilon\boldsymbol{s}) \mid_{\varepsilon=0} \tag{3.8}$$

exists.

Theorem 3.1. *Functional $\boldsymbol{x}(t;\boldsymbol{u})$ defined by (3.6) is Gateaux differentiable with*

$$\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u}) = \int_{t_0}^{t}\nabla\boldsymbol{x}(t;\boldsymbol{u})(\tau)^T\boldsymbol{s}(\tau)d\tau, \tag{3.9}$$

where

$$\nabla\boldsymbol{x}(t;\boldsymbol{u})(\tau) \quad = \quad \boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(\tau;\boldsymbol{u}),\boldsymbol{u}(\tau))^T\Phi(t,\tau)^T,\ t_0 \le \tau \le t \tag{3.10}$$

and $\Phi(t,\tau)$ is a continuous transition matrix function.

Proof. Integrating (3.1) from t_0 to t with $\boldsymbol{u}+\varepsilon\boldsymbol{s}$ given, we have

$$\boldsymbol{x}(t;\boldsymbol{u}+\varepsilon\boldsymbol{s}) \quad = \quad \boldsymbol{x}(t_0)+\int_{t_0}^{t}\boldsymbol{f}(\boldsymbol{x}(\tau;\boldsymbol{u}+\varepsilon\boldsymbol{s}),\boldsymbol{u}(\tau)+\varepsilon\boldsymbol{s}(\tau))d\tau. \tag{3.11}$$

Differentiating (3.11) w. r. t ε and letting $\varepsilon = 0$, and then differentiating it w. r. t. t, we finally obtain

$$\begin{aligned}&\frac{d}{dt}\frac{d}{d\varepsilon}\boldsymbol{x}(t;\boldsymbol{u}+\varepsilon\boldsymbol{s}) \mid_{\varepsilon=0}\\ &= \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))\frac{d}{d\varepsilon}\boldsymbol{x}(t;\boldsymbol{u}+\varepsilon\boldsymbol{s}) \mid_{\varepsilon=0} +\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))\boldsymbol{s}(t).\end{aligned} \tag{3.12}$$

The equation (3.12) is a time-varying linear differential equation, where the unknown variable is

$$\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u}) \quad = \quad \frac{d}{d\varepsilon}\boldsymbol{x}(t;\boldsymbol{u}+\varepsilon\boldsymbol{s}) \mid_{\varepsilon=0}.$$

Its solution is given by

$$\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u}) \quad = \quad \int_{t_0}^{t}\Phi(t,\tau)\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(\tau,\boldsymbol{u}),\boldsymbol{u}(\tau))\boldsymbol{s}(\tau)d\tau \tag{3.13}$$

where $\Phi(t,\tau)$ is a continuous transition matrix function that has properties

$$\frac{\partial}{\partial t}\Phi(t,\tau) = \boldsymbol{f_x}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))\Phi(t,\tau),\ t_0 \le \tau \le t$$
$$\Phi(\tau,\tau) = I.$$

According to (3.4), $\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u})$ can be rewritten in the inner product form

$$\delta_{\boldsymbol{s}}\boldsymbol{x}(t;\boldsymbol{u}) = < \nabla\boldsymbol{x}(t;\boldsymbol{u}), \boldsymbol{s} >$$

where

$$\nabla\boldsymbol{x}(t;\boldsymbol{u})(\tau) = \boldsymbol{f_u}(\boldsymbol{x}(\tau;\boldsymbol{u}),\boldsymbol{u}(\tau))^T\Phi(t,\tau)^T,\ t_0 \le \tau \le t. \tag{3.14}$$

□

Theorem 3.2. *By defining* $\nabla_{\boldsymbol{u}}\boldsymbol{x}(t;\boldsymbol{u}) = \nabla\boldsymbol{x}(t;\boldsymbol{u})(t)$*, the gradient of objective function* $F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))$ *with respect to* $\boldsymbol{u}(t)$ *at time* t *is*

$$\nabla_{\boldsymbol{u}}F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t)) = \boldsymbol{f_u}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T. \tag{3.15}$$

Proof. By chain rule,

$$\nabla_{\boldsymbol{u}}F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}) = (F_{\boldsymbol{x}}(\boldsymbol{x}(t,\boldsymbol{u}),\boldsymbol{u})\nabla_{\boldsymbol{u}}\boldsymbol{x}(t;\boldsymbol{u}))^T + (F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}))^T. \tag{3.16}$$

From (3.10),

$$\begin{aligned}\nabla\boldsymbol{x}(t;\boldsymbol{u})(t) &= \boldsymbol{f_u}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\Phi(t,t)^T\\ &= \boldsymbol{f_u}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T.\end{aligned} \tag{3.17}$$

By substituting equation (3.17) into equation (3.16), we obtain

$$\nabla_{\boldsymbol{u}}F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}) = \boldsymbol{f_u}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T. \tag{3.18}$$

□

Thus, the dynamic control (3.7) becomes

$$\dot{\boldsymbol{u}}(t) = -\alpha\{\boldsymbol{f_u}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t,\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\}. \tag{3.19}$$

This control law is called the direct gradient descent control [6, 7].

Example 3.1. Consider the nonlinear control system

$$\dot{x_1} = x_1x_2, \tag{3.20}$$

$$\dot{x_2} = -x_2 - 2x_1^2 + x_2u. \tag{3.21}$$

It is easy to show that by linearization, system (3.20)-(3.21) can not be stabilized asymptotically.

Let the performance index $F(\boldsymbol{x},\boldsymbol{u}) = \frac{1}{2}(x_1^2 + x_2^2 + u^2)$. By applying the direct gradient descent control to nonlinear system (3.20)-(3.21) we have the extended system

$$\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{u}\end{pmatrix} = \begin{pmatrix}x_1x_2\\ -x_2 - 2x_1^2 + x_2u\\ -\alpha x_2^2 - \alpha u\end{pmatrix}. \tag{3.22}$$

Then, we find the value of parameter α by Center Manifold Theory. Let $x_1 = z$ and $\mathbf{y} = [y_1 \;\; y_2]^t = [x_2 \;\; u]^t$. The equation (3.20)-(3.21) can be written as follows.

$$\dot{z} = zy_1 = Az + f(z, y), \tag{3.23}$$

$$\dot{y} = \begin{pmatrix} -1 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -2z^2 + y_1 y_2 \\ -\alpha y_1^2 \end{pmatrix}$$

$$= By + g(z, y). \tag{3.24}$$

Let $\mathbf{y} = \mathbf{h}(z)$ as a center manifold for that system and approximate it by

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ u \end{pmatrix} = \begin{pmatrix} h_1(z) \\ h_2(z) \end{pmatrix} \tag{3.25}$$

$$= \begin{pmatrix} a_1 z^2 + a_2 z^3 + \dots \\ b_1 z^2 + b_2 z^3 + \dots \end{pmatrix}. \tag{3.26}$$

For finding coefficients a_i, b_i we solve the equation

$$\dot{\mathbf{y}} = Dh(z)\dot{z}, \tag{3.27}$$

$$B\mathbf{y} + g(z, \mathbf{y}) = Dh(z)[Az + f(z, \mathbf{y})] \tag{3.28}$$

i. e.

$$\begin{pmatrix} -1 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -2z_2 + y_1 y_2 \\ -\alpha y_1^2 \end{pmatrix} \tag{3.29}$$

$$= \begin{pmatrix} 2a_1 z + 3a_2 z^2 + \dots \\ 2b_1 z + 3b_2 z^2 + \dots \end{pmatrix} \begin{pmatrix} zy_1 \end{pmatrix} \tag{3.30}$$

and we obtained $a_1 = -2, \;\; b_1 = -4$ for $\alpha > 0$, and

$$\mathbf{y} = \begin{pmatrix} -2z^2 + \mathcal{O}(x^4) \\ -4z^2 + \mathcal{O}(x^4) \end{pmatrix}. \tag{3.31}$$

Thus, vector field in center manifold is

$$\dot{z} = -2z^3 + (O)(x^5). \tag{3.32}$$

According to center manifold theory, the behavior of the system (3.20)-(3.21) can be determined from behavior of the system in equation (3.32). Thus, because the solution of system in equation (3.32) is asymptotically stable then the solution of system in equation (3.20)-(3.21) is asymptotically stable. Simulation result is shown in Fig. 3.

3.1. Direct gradient descent control as a dynamic feedback control. Consider the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \;\; \boldsymbol{x} \in \mathcal{R}^n, \;\; \boldsymbol{u} \in \mathcal{R}^r, \tag{3.33}$$

where A and B are real matrices of dimension $(n \times n)$ and $(n \times r)$.

In the following, we apply DGDC to stabilize the system (3.33) asymptotically.

Case I: Unforced system is stable

Consider the linear system (3.33) and assume that $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is stable.

Define performance index:

$$\hat{F}(\boldsymbol{x}, \boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{u}^T R \boldsymbol{u}\right) \tag{3.34}$$

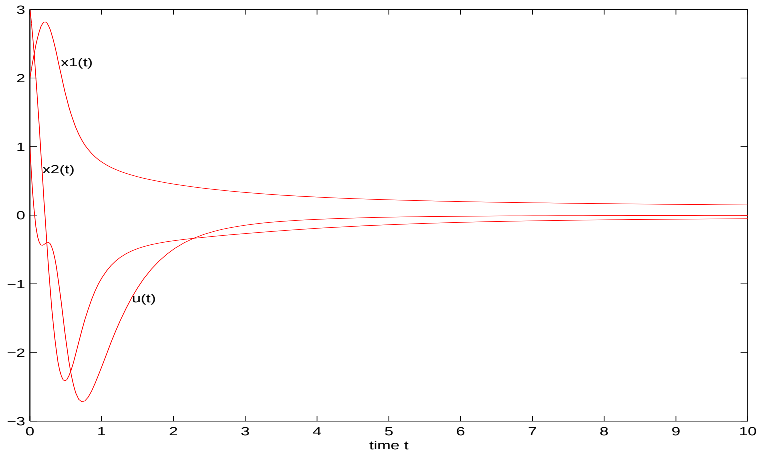

Figure 3. Gradient Descent Control

where $R > 0$, $Q > 0$ and

$$\boldsymbol{x}^T Q A \boldsymbol{x} \leq 0. \tag{3.35}$$

(Since $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is stable, there exists Q, for instance $Q = I$, such that $\boldsymbol{x}^T Q A \boldsymbol{x} \leq 0$.)

The direct gradient descent control for system (3.33) with performance index (3.34) is

$$\dot{\boldsymbol{u}} = -\alpha \left(B^T Q \boldsymbol{x} + R \boldsymbol{u} \right). \tag{3.36}$$

Take $\alpha = R^{-1}$. Then we have the extended system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{3.37}$$

$$\dot{\boldsymbol{u}} = -R^{-1} B^T Q \boldsymbol{x} - \boldsymbol{u}. \tag{3.38}$$

In this section, we investigate the condition when the extended system (3.37)-(3.38) becomes asymtotically stable. First, consider the objective function (3.34) as a candidate Lyapunov function for extended system (3.37)-(3.38). Time derivative of performance index (3.34) along the trajectory of extended system (3.37)-(3.38) is

$$\dot{\hat{F}}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}^T Q A \boldsymbol{x} - \boldsymbol{u}^T R \boldsymbol{u} \leq 0. \tag{3.39}$$

From (3.35) we have

$$\dot{\hat{F}}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}^T Q A \boldsymbol{x} - \boldsymbol{u}^T R \boldsymbol{u} < 0, \ \forall \ \boldsymbol{u} \neq 0. \tag{3.40}$$

But $\dot{\hat{F}}(\boldsymbol{x}, \boldsymbol{u}) \leq 0$ if $\boldsymbol{u} = 0$. To obtain asymptotical stability of the extended system (3.37)-(3.38), the value of $\dot{\hat{F}}(\boldsymbol{x}, \boldsymbol{u})$ must be less than zero for all nonzero $\boldsymbol{x}$ and $\boldsymbol{u}$. For this we need a condition: if $\boldsymbol{u}$ becomes zero then $\boldsymbol{x}$ becomes zero.

Below we investigate when that condition is satisfied. From (3.34), (3.39) and LaSalle-Yoshizawa Theorem [4] we have:

$$\lim_{t \to \infty} \dot{\hat{F}}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = 0.$$

Hence:

$$\lim_{t\to\infty} \boldsymbol{u}(t) = 0.$$

According to Barbalat Lemma [9]:

$$\lim_{t\to\infty} \dot{\boldsymbol{u}}(t) = 0.$$

Let us assume that for $t > t_1$, $\boldsymbol{u}(t) << 0$ and $\dot{\boldsymbol{u}}(t) << 0$. Then we investigate the behavior of the extended system (3.37)-(3.38) for $t > t_1$. Thus for $t > t_1$, equations (3.37)-(3.38) can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} \\ 0 &= -R^{-1}B^TQ\boldsymbol{x}. \end{aligned} \tag{3.41}$$

If the system (3.41) is asymptotically stable, then the system (3.37)-(3.38) becomes asymptotically stable.

Example 3.2. Let the linear control system be given as follows.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{3.42}$$

where

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.43}$$

is stable. By performance index

$$F(\boldsymbol{x}, u) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x} + \frac{1}{2}ru^2 \tag{3.44}$$

we have the direct gradient descent control:

$$\dot{u} = -\frac{1}{r}B^TQ\boldsymbol{x} - u = -\frac{1}{r}x_1 - u, \tag{3.45}$$

and extended system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 0 \\ -\frac{1}{r} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \tag{3.46}$$

with

$$\dot{F}(\boldsymbol{x}, u) = \boldsymbol{x}^T A\boldsymbol{x} - ru^2 \leq 0. \tag{3.47}$$

Below, we give steps to analyze the behavior of the system in equation (3.46).

- $F(\boldsymbol{x}(t), u(t))$ is a decreasing function about time t. Then
$$\lim_{t\to\infty} \dot{F}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = 0.$$
- Then $u(t) \to 0$ if $t \to \infty$ and $\boldsymbol{x}(t)$ bounded.
- Since $\boldsymbol{x}(t)$ is bounded, then $\ddot{u}(t) = -\frac{1}{r}(a_1+1)x_1 + \frac{a_2}{r}x_2 + (1-\frac{1}{r})u$ is bounded.
- Barbalat Lemma: $\dot{u}(t) \to 0$ if $t \to \infty$.

- Assume that for $t \geq t_1$, $\boldsymbol{u}(t) << 0$ and $\dot{\boldsymbol{u}}(t) << 0$, then equation (3.46) becomes

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ 0 &= -\frac{1}{r} x_1. \end{aligned} \tag{3.48}$$

- $x_1(t) = 0$, $t \geq t_1$.
- Since $\ddot{\boldsymbol{x}}(t)$ is bounded, then $\dot{x}_1(t) \to 0$ if $t \to \infty$.
- For $t \geq t_1$ equation (3.48) becomes

$$\begin{aligned} 0 &= 0 + a_2 x_2 \\ \dot{x_2} &= 0 + b_2 x_2 \end{aligned} \tag{3.49}$$

and finally we have $x_2(t) = 0$, $t \geq t_1$.

By these steps, Example 3.2 shows us that if $\boldsymbol{u}$ becomes zero then $\boldsymbol{x}$ becomes zero. It means that the extended system (3.46) becomes asymptotically stable about the equilibrium point $(0, 0)$.

Based on above analyze for guaranteing the stability (asymptotically) of extended system (3.37)-(3.38) we need the following assumption. This assumption we refer to equation (3.48).

Assumption 3.1. Subsystem

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} \\ 0 &= -R^{-1}B^T Q\boldsymbol{x}. \end{aligned} \tag{3.50}$$

is asymptotically stable.

Thus, by Assumption 3.1, the condition: if $\boldsymbol{u}$ becomes zero then $\boldsymbol{x}$ becomes zero is satisfied. Hence

$$\dot{F}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}^T QA\boldsymbol{x} - \boldsymbol{u}^T R\boldsymbol{u} < 0, \ \forall \ \boldsymbol{u} \neq 0 \ and \ \boldsymbol{x} \neq 0. \tag{3.51}$$

It means that global asymptotical stability of the extended system (3.37)-(3.38) is achieved. We state the above result in the following Proposition.

Proposition 3.1. *Consider the linear system (3.33) and assume that $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is stable. Let performance index be defined as (3.34). By applying direct gradient descent control (3.19) to system (3.33) and take $\alpha = R_1^{-1}$, we get the extended system (3.37)-(3.38). If Assumption 3.1 is satisfied then the extended system (3.37)-(3.38) becomes globally asymptotically stable.*

Remark 3.1. Consider $B^T Q\boldsymbol{x}$ as an output of the system. Assume that system

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \boldsymbol{y} &= B^T Q\boldsymbol{x} \end{aligned}$$

is zero-state observable [1] and unforced system, i. e. $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is stable. Then the extended system (3.37)-(3.38) becomes globally asymptotically stable.

Case II: The stability of unforced system is unknown

Consider the extended system which is formed by (3.33) and (3.36), i. e.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{3.52}$$

$$\dot{\boldsymbol{u}} = -\alpha B^T Q\boldsymbol{x} - \alpha R\boldsymbol{u}. \tag{3.53}$$

Let

$$\bar{A} = \begin{bmatrix} A & B \\ -\alpha B^T Q & -\alpha R \end{bmatrix}. \tag{3.54}$$

The extended system (3.52)-(3.53) becomes asymptotically stable if there exist α, Q and R such that matrix $\bar{A}$ becomes Hurwitz. According to Proposition 2.1, values of α, Q and R can be found if system (3.52) is controllable.

4. Conclusions

We have studied the dynamic feedback control for stabilization of linear control system. In linear system have been shown that the dynamic state feedback control exists if static state feedback control exists when dynamic output feedback control does not always exists even if static output feedback control exists. By assuming the unforced system is stable and condition: if $\boldsymbol{u}$ becomes zero then $\boldsymbol{x}$ becomes zero, the direct gradient descent control as a dynamic feedback control stabilizes the linear system asymptotically.

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