**Fuzzy QS-Algebras with Interval-Valued Membership Functions**

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**Abstract.** In this note the notion of interval-valued fuzzy QS-algebra (briefly, i-v fuzzy QS-algebra), as well as the i-v level and strong i-v level QS-subalgebra is introduced. Several theorems which determine the relationship between these notions and QS-subalgebras are stated and proved. The images and inverse images of i-v fuzzy QS-subalgebras are defined, and how the homomorphic images and inverse images of an i-v fuzzy QS-subalgebra become i-v fuzzy QS-algebras is studied as well.

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**1. Introduction**

In 1966, Imai and Iseki [7] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers, Ahn and Kim introduced the notion of Q-algebras [11], which is a generalization of BCH/BCI/BCK-algebras. In [1], Ahn and Kim introduced the notion of QS-algebras which is a generalization of Q-algebras.

The concept of a fuzzy set, was introduced in [12]. In [13], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. He constructed a method of approximate inference using his i-v fuzzy sets. Biswas [2], defined interval-valued fuzzy subgroups and Hong et al. applied the notion of interval-valued fuzzy sets to BCI-algebras [4].

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In the present paper, we use the notion of interval-valued fuzzy set and introduce the concept of interval-valued fuzzy QS-subalgebras (briefly i-v fuzzy QS-subalgebras) of a QS-algebra, and we study some of their properties. Among other results, we prove that every QS-subalgebra of a QS-algebra $X$ can be realized as an i-v level QS-subalgebra of an i-v fuzzy QS-subalgebra of $X$. We also obtain some related results which have been mentioned in the abstract.

2. Preliminary notes

**Definition 2.1.** [1] A QS-algebra is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * z = (x * z) * y$,
4. $(x * y) * (x * z) = z * y$,

for all $x, y, z \in X$.

In $X$ we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$

**Example 2.1.** [1] Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} = \{nz \mid z \in \mathbb{Z}\}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are both QS-algebras, where "-" is the usual subtraction of integers. Also $(\mathbb{R}; -, 0)$ and $(\mathbb{C}; -, 0)$ are QS-algebras where $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers and "-" is the usual subtraction of real (complex) numbers.

**Proposition 2.1.** [1] Let $X$ be a QS-algebra. Then for any $x, y$ and $z$ in $X$, the following relations hold:

1. $x \leq y$ implies $z * y \leq z * x$,
2. $x \leq y$ and $y \leq z$ imply $x \leq z$,
3. $x * y \leq z$ implies $x * z \leq y$,
4. $(x * z) * (y * z) \leq x * y$,
5. $x \leq y$ implies $x * z \leq y * z$,
6. $0 * (0 * x) = 0 * x$.

**Definition 2.2.** A non-empty subset $S$ of a QS-algebra $X$ is called a subalgebra of $X$ if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \rightarrow Y$ of QS-algebras is called a QS-homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

We now review some fuzzy logic concepts (see [12]). Let $X$ be a set. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let $f$ be a mapping from the set $X$ to the set $Y$ and let $B$ be a fuzzy set in $Y$ with membership function $\mu_B$. The inverse image of $B$, denoted $f^{-1}(B)$, is the fuzzy set in $X$ with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$. Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A$. Then the image of $A$, denoted by $f(A)$, is the fuzzy set in $Y$ such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$
A fuzzy set $A$ in the $QS$-algebra $X$ with the membership function $\mu_A$ is said to have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t).$$

An interval-valued fuzzy set (briefly, i-v fuzzy set) $A$ defined on $X$ is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \forall x \in X.$$ 

Briefly, it is denoted by $A = [\mu_A^L, \mu_A^U]$ where $\mu_A^L$ and $\mu_A^U$ are any two fuzzy sets in $X$ such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$ then $\overline{\mu}_A(x) = [c, c]$ is in $D[0,1]$. Thus $\overline{\mu}_A(x) \in D[0,1]$, for all $x \in X$. Therefore the i-v fuzzy set $A$ is given by

$$A = \{(x, \overline{\mu}_A(x))\}, \forall x \in X$$

where

$$\overline{\mu}_A : X \rightarrow D[0,1].$$

Now we define the refined minimum (briefly, rmin) and order ”$\leq$” on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$\text{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}],$$

$$D_1 \leq D_2 \iff a_1 \leq a_2 \land b_1 \leq b_2.$$ 

Similarly we can define $\geq$ and $=$.

**Definition 2.3.** [3] Let $\mu$ be a fuzzy set in a $QS$-algebra $X$. Then $\mu$ is called a fuzzy $QS$-subalgebra ($QS$-algebra) of $X$ if

$$\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in X$.

**Proposition 2.2.** [3] Let $f$ be a $QS$-homomorphism from $X$ into $Y$ and $G$ be a fuzzy $QS$-subalgebra of $Y$ with the membership function $\mu_G$. Then the inverse image $f^{-1}(G)$ of $G$ is a fuzzy $QS$-subalgebra of $X$.

**Proposition 2.3.** [3] Let $f$ be a $QS$-homomorphism from $X$ onto $Y$ and $D$ be a fuzzy $QS$-subalgebra of $X$ with the sup property. Then the image $f(D)$ of $D$ is a fuzzy $QS$-subalgebra of $Y$.

### 3. Interval-valued fuzzy $QS$-algebra

From now on $X$ is a $QS$-algebra, unless otherwise is stated.

**Definition 3.1.** An i-v fuzzy set $A$ in $X$ is called an interval-valued fuzzy $QS$-subalgebras (briefly i-v fuzzy $QS$-subalgebra) of $X$ if

$$\overline{\mu}_A(x \ast y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\},$$

for all $x, y \in X$. 
**Example 3.1.** Let \( X = \{0, 1, 2\} \) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a \(QS\)-algebra, but not a \(BCH/BCI/BCK\)-algebra.

Define \(\mu_A\) as:

\[
\mu_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{otherwise.} \end{cases}
\]

It is easy to check that \(A\) is an \(i-v\) fuzzy \(QS\)-subalgebra of \(X\).

**Lemma 3.1.** If \(A\) is an \(i-v\) fuzzy \(QS\)-subalgebra of \(X\), then

\[
\overline{\mu}_A(0) \geq \overline{\mu}_A(x)
\]

for all \(x \in X\).

**Proof.** For all \(x \in X\), we have

\[
\overline{\mu}_A(0) = \overline{\mu}_A(x * x) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(x)\}
\]

\[
= \text{rmin}\{[\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(x), \mu^U_A(x)]\}
\]

\[
= [\mu^L_A(x), \mu^U_A(x)] = \overline{\mu}_A(x).
\]

\[
\square
\]

**Proposition 3.1.** Let \(A\) be an \(i-v\) fuzzy \(QS\)-subalgebra of \(X\), and let \(n \in \mathbb{N}\). Then

(i) \(\overline{\mu}_A(\prod_{n} x * x) \geq \overline{\mu}_A(x)\), for any odd number \(n\),

(ii) \(\overline{\mu}_A(\prod_{n} x * x) = \overline{\mu}_A(0)\), for any even number \(n\).

**Proof.** We proved by induction. Let \(x \in X\) and assume that \(n\) is odd. Then \(n = 2k - 1\) for some positive integer \(k\). The definition and the above lemma imply that \(\overline{\mu}_A(x * x) = \overline{\mu}_A(0) \geq \overline{\mu}_A(x)\). Now suppose that \(\overline{\mu}_A(\prod_{n} x * x) \geq \overline{\mu}_A(x)\). Then by assumption

\[
\overline{\mu}_A(\prod_{2(k+1)-1} x * x) = \overline{\mu}_A(\prod_{2k+1} x * x)
\]

\[
= \overline{\mu}_A(\prod_{2k-1} x * (x * (x * x)))
\]

\[
= \overline{\mu}_A(\prod_{2k-1} x * x)
\]

\[
\geq \overline{\mu}_A(x).
\]

This proves (i), and similarly we can prove (ii). \(\square\)
Theorem 3.1. Let $A$ be an $i$-$v$ fuzzy $QS$-subalgebra of $X$. If there exists a sequence $\{x_n\}$ in $X$, such that
$$\lim_{n \to \infty} \overline{\mu}_A(x_n) = [1, 1].$$
Then $\overline{\mu}_A(0) = [1, 1]$.

Proof. By the above lemma we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$, for all $x \in X$, thus $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$, for every positive integer $n$. Consider the inequality
$$[1, 1] \geq \overline{\mu}_A(0) \geq \lim_{n \to \infty} \overline{\mu}_A(x_n) = [1, 1].$$
Hence $\overline{\mu}_A(0) = [1, 1]$. \hfill $\Box$

Theorem 3.2. An $i$-$v$ fuzzy set $A = [\mu^L_A, \mu^U_A]$ in $X$ is an $i$-$v$ fuzzy $QS$-subalgebra of $X$ if and only if $\mu^L_A$ and $\mu^U_A$ are fuzzy $QS$-subalgebras of $X$.

Proof. Let $\mu^L_A$ and $\mu^U_A$ be fuzzy $QS$-subalgebras of $X$ and $x, y \in X$. Observe
$$\overline{\mu}_A(x * y) = [\mu^L_A(x * y), \mu^U_A(x * y)]$$
$$\geq \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}$$
$$= \min\{\mu^L_A(x), \mu^U_A(y)\}, \min\{\mu^U_A(x), \mu^L_A(y)\}$$
Therefore $\mu^L_A(x * y) \geq \min\{\mu^L_A(x), \mu^L_A(y)\}$ and $\mu^U_A(x * y) \geq \min\{\mu^U_A(x), \mu^L_A(y)\}$, whence we get that $\mu^L_A$ and $\mu^U_A$ are fuzzy $QS$-subalgebras of $X$. \hfill $\Box$

Theorem 3.3. Let $A_1$ and $A_2$ be $i$-$v$ fuzzy $QS$-subalgebras of $X$. Then $A_1 \cap A_2$ is an $i$-$v$ fuzzy $QS$-subalgebra of $X$.

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and $A_2$, since $A_1$ and $A_2$ are $i$-$v$ fuzzy $QS$-subalgebras of $X$, by the above theorem we have:
$$\overline{\mu}_{A_1 \cap A_2}(x * y) = [\mu^L_{A_1 \cap A_2}(x * y), \mu^U_{A_1 \cap A_2}(x * y)]$$
$$= \min\{\mu^L_{A_1}(x * y), \mu^L_{A_2}(x * y)\}, \min\{\mu^U_{A_1}(x * y), \mu^U_{A_2}(x * y)\}$$
$$\geq \min\{\mu^L_{A_1 \cap A_2}(x), \mu^L_{A_1 \cap A_2}(y)\}, \min\{\mu^U_{A_1 \cap A_2}(x), \mu^U_{A_1 \cap A_2}(y)\}$$
Therefore
$$\overline{\mu}_{A_1 \cap A_2}(x * y) \geq \min\{\mu^L_{A_1 \cap A_2}(x), \mu^L_{A_1 \cap A_2}(y)\}$$
$$\mu^U_{A_1 \cap A_2}(x * y) \geq \min\{\mu^U_{A_1 \cap A_2}(x), \mu^L_{A_1 \cap A_2}(y)\}$$
whence we get that $\mu^L_{A_1 \cap A_2}$ and $\mu^U_{A_1 \cap A_2}$ are fuzzy $QS$-subalgebras of $X$. \hfill $\Box$

Corollary 3.8. Let $\{A_i|i \in \Lambda\}$ be a family of $i$-$v$ fuzzy $QS$-subalgebras of $X$. Then $\bigcap_{i \in \Lambda} A_i$ is also an $i$-$v$ fuzzy $QS$-subalgebra of $X$. 

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Definition 3.2. Let $A$ be an i-v fuzzy set in $X$ and $[\delta_1, \delta_2] \in D[0,1]$. Then the i-v level $QS$-subalgebra $U(A; [\delta_1, \delta_2])$ of $A$ and strong i-v level $QS$-subalgebra $U(A; >\delta, [\delta_1, \delta_2])$ of $X$ are defined as follows:

$$U(A; [\delta_1, \delta_2]) := \{ x \in X \mid \overline{\mu}_A(x) \geq [\delta_1, \delta_2] \},$$

$$U(A; >\delta, [\delta_1, \delta_2]) := \{ x \in X \mid \overline{\mu}_A(x) > [\delta_1, \delta_2] \}. $$

Theorem 3.4. Let $A$ be an i-v fuzzy $QS$-subalgebra of $X$ and let $B$ be the closure of the image of $\mu_A$. Then the following conditions are equivalent:

(i) $A$ is an i-v fuzzy $QS$-subalgebra of $X$.

(ii) For all $[\delta_1, \delta_2] \in \mathcal{A}(\mu_A)$, the nonempty i-v level subset $U(A; [\delta_1, \delta_2])$ of $A$ is a $QS$-subalgebra of $X$.

(iii) For all $[\delta_1, \delta_2] \in \mathcal{A}(\mu_A) \setminus B$, the nonempty strong i-v level subset $U(A; >\delta, [\delta_1, \delta_2])$ of $A$ is a $QS$-subalgebra of $X$.

(iv) For all $[\delta_1, \delta_2] \in D[0,1]$, the nonempty strong i-v level subset $U(A; >\delta, [\delta_1, \delta_2])$ of $A$ is a $QS$-subalgebra of $X$.

(v) For all $[\delta_1, \delta_2] \in D[0,1]$, the nonempty i-v level subset $U(A; [\delta_1, \delta_2])$ of $A$ is a $QS$-subalgebra of $X$.

Proof. (i) $\implies$ (iv). Let $A$ be an i-v fuzzy $QS$-subalgebra of $X$, $[\delta_1, \delta_2] \in D[0,1]$ and $x, y \in U(A; <\delta_1, [\delta_1, \delta_2])$, then we have

$$\overline{\mu}_A(x * y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} > \min\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2],$$

and thus $x * y \in U(A; >\delta_1, [\delta_1, \delta_2])$. Hence $U(A; >\delta_1, [\delta_1, \delta_2])$ is a $QS$-subalgebra of $X$.

(iv) $\implies$ (iii). It is clear.

(iii) $\implies$ (ii). If $[\delta_1, \delta_2] \in \mathcal{A}(\mu_A)$, then $U(A; [\delta_1, \delta_2])$ is nonempty, since

$$U(A; [\delta_1, \delta_2]) = \bigcap_{[\delta_1, \delta_2] > [\alpha_1, \alpha_2]} U(A; >\delta, [\delta_1, \delta_2]),$$

where $[\alpha_1, \alpha_2] \in \mathcal{A}(\mu_A) \setminus B$. Then by (iii) and Corollary 3.7, $U(A; [\delta_1, \delta_2])$ is a $QS$-subalgebra of $X$.

(ii) $\implies$ (v). Let $[\delta_1, \delta_2] \in D[0,1]$ and $U(A; [\delta_1, \delta_2])$ be nonempty. Suppose $x, y \in U(A; [\delta_1, \delta_2])$. Let $[\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\}$, it is clear that

$$[\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\} \geq \{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2].$$

Thus $x, y \in U(A; [\beta_1, \beta_2])$ and $[\beta_1, \beta_2] \in \mathcal{A}(\mu_A)$, by (ii) $U(A; [\beta_1, \beta_2])$ is a $QS$-subalgebra of $X$, hence $x * y \in U(A; [\beta_1, \beta_2])$. Then we have

$$\overline{\mu}_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2] \geq [\delta_1, \delta_2].$$

Therefore $x * y \in U(A; [\delta_1, \delta_2])$. Then $U(A; [\delta_1, \delta_2])$ is a $QS$-subalgebra of $X$.

(v) $\implies$ (i). Assume that the nonempty set $U(A; [\delta_1, \delta_2])$ is a $QS$-subalgebra of $X$, for every $[\delta_1, \delta_2] \in D[0,1]$. In the contrary, let $x_0, y_0 \in X$ be such that

$$\overline{\mu}_A(x_0 * y_0) < \min\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}.$$ 

Let $\overline{\mu}_A(x_0) = [\gamma_1, \gamma_2]$, $\overline{\mu}_A(y_0) = [\gamma_3, \gamma_4]$ and $\overline{\mu}_A(x_0 * y_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] < \min\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = \min\{\gamma_1, \gamma_3\} \min\{\gamma_2, \gamma_4\}.$$ 

So $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$. 

Consider
\[ [\lambda_1, \lambda_2] = \frac{1}{2} \overline{\mu}_A(x_0 \ast y_0) + \min\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}. \]

We find that
\[ [\lambda_1, \lambda_2] = \frac{1}{2} ([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}]) \]
\[ = [\frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}), \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\})]. \]

Therefore
\[ \min\{\gamma_1, \gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}) > \delta_1, \]
\[ \min\{\gamma_2, \gamma_4\} > \lambda_2 = \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\}) > \delta_2. \]

Hence
\[ [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2] = [\delta_1, \delta_2] = \overline{\mu}_A(x_0 \ast y_0), \]
so that \( x_0 \ast y_0 \not\in U(A; [\delta_1, \delta_2]) \) which is a contradiction, since
\[ \overline{\mu}_A(x_0) = [\gamma_1, \gamma_2] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2], \]
\[ \overline{\mu}_A(y_0) = [\gamma_3, \gamma_4] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2], \]

imply that \( x_0, y_0 \in U(A; [\delta_1, \delta_2]). \) Thus \( \overline{\mu}_A(x \ast y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \) for all \( x, y \in X, \) which completes the proof. \( \square \)

**Theorem 3.5.** Each QS-subalgebra of \( X \) is an i-v level QS-subalgebra of an i-v fuzzy QS-subalgebra of \( X \).

**Proof.** Let \( Y \) be a QS-subalgebra of \( X \), and \( A \) be an i-v fuzzy set on \( Y \) defined by
\[ \overline{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in Y \\ [0, 0] & \text{otherwise} \end{cases} \]
where \( \alpha_1, \alpha_2 \in [0, 1] \) with \( \alpha_1 < \alpha_2 \). It is clear that \( U(A; [\alpha_1, \alpha_2]) = Y \). Let \( x, y \in X \). If \( x, y \in Y \), then \( x \ast y \in Y \) and therefore
\[ \overline{\mu}_A(x \ast y) = [\alpha_1, \alpha_2] = \min\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \]

If \( x, y \not\in Y \), then \( \overline{\mu}_A(x) = [0, 0] = \overline{\mu}_A(y) \) and so
\[ \overline{\mu}_A(x \ast y) \geq [0, 0] = \min\{[0, 0], [0, 0]\} = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \]

If \( x \in Y \) and \( y \not\in Y \), then \( \overline{\mu}_A(x) = [\alpha_1, \alpha_2] \) and \( \overline{\mu}_A(y) = [0, 0] \). Thus
\[ \overline{\mu}_A(x \ast y) \geq [0, 0] = \min\{[\alpha_1, \alpha_2], [0, 0]\} = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \]
Similarly, if \( y \in Y \) and \( x \not\in Y \), then \( \overline{\mu}_A(x \ast y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \). Therefore \( A \) is an i-v fuzzy QS-subalgebra of \( X \). \( \square \)

**Theorem 3.6.** Let \( Y \) be a subset of \( X \) and \( A \) be the i-v fuzzy set on \( X \) which is given in the proof of Theorem 3.5 If \( A \) is an i-v fuzzy QS-subalgebra of \( X \), then \( Y \) is a QS-subalgebra of \( X \).
Proof. Let $A$ be an $i,v$ fuzzy $QS$-subalgebra of $X$, and $x,y \in Y$. Then $\overline{p}_A(x) = [\alpha_1, \alpha_2] = \overline{p}_A(y)$, thus

$$\overline{p}_A(x * y) \geq \text{rmin}\{\overline{p}_A(x), \overline{p}_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

which implies that $x * y \in Y$. □

Theorem 3.7. If $A$ is an $i,v$ fuzzy $QS$-subalgebra of $X$, then the set

$$X_{\overline{p}_A} := \{x \in X \mid \overline{p}_A(x) = \overline{p}_A(0)\},$$

is a $QS$-subalgebra of $X$.

Proof. Let $x,y \in X_{\overline{p}_A}$. Then $\overline{p}_A(x) = \overline{p}_A(0) = \overline{p}_A(y)$, and so

$$\overline{p}_A(x * y) \geq \text{rmin}\{\overline{p}_A(x), \overline{p}_A(y)\} = \text{rmin}\{\overline{p}_A(0), \overline{p}_A(0)\} = \overline{p}_A(0).$$

By Lemma 3.1, we get that $\overline{p}_A(x * y) = \overline{p}_A(0)$ which means that $x * y \in X_{\overline{p}_A}$. □

Theorem 3.8. Let $N$ be an $i,v$ fuzzy subset of $X$ defined by

$$\overline{p}_N(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in N \\ [\beta_1, \beta_2] & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$. Then $N$ is an $i,v$ fuzzy $QS$-subalgebra if and only if $N$ is a $QS$-subalgebra of $X$. Moreover, in this case $X_{\overline{p}_N} = N$.

Proof. Let $N$ be an $i,v$ fuzzy $QS$-subalgebra. Let $x,y \in X$ be such that $x,y \in N$. Then

$$\overline{p}_N(x * y) \geq \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2],$$

and so $x * y \in N$.

Conversely, suppose that $N$ is a $QS$-subalgebra of $X$, and let $x,y \in X$.

(i) If $x,y \in N$ then $x * y \in N$, thus

$$\overline{p}_N(x * y) = [\alpha_1, \alpha_2] = \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\}.$$ (ii) If $x \not\in N$ or $y \not\in N$, then

$$\overline{p}_N(x * y) \geq [\beta_1, \beta_2] = \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\}.$$ This shows that $N$ is an $i,v$ fuzzy $QS$-subalgebra.

Moreover, we have

$$X_{\overline{p}_N} := \{x \in X \mid \overline{p}_N(x) = \overline{p}_N(0)\} = \{x \in X \mid \overline{p}_N(x) = [\alpha_1, \alpha_2]\} = N.$$ □

Definition 3.3. [2] Let $f$ be a mapping from the set $X$ into a set $Y$. Let $B$ be an $i,v$ fuzzy set in $Y$. Then the inverse image of $B$, denoted by $f^{-1}[B]$, is the $i,v$ fuzzy set in $X$ with the membership function given by $\overline{p}_{f^{-1}[B]}(x) = \overline{p}_B(f(x))$, for all $x \in X$.

Lemma 3.2. [2] Let $f$ be a mapping from the set $X$ into the set $Y$. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be $i,v$ fuzzy sets in $X$ and $Y$ respectively. Then

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,
(ii) $f(m) = [f(m^L), f(m^U)].$
Proposition 3.2. Let $f$ be a $QS$-homomorphism from $X$ into $Y$ and $G$ be an i-v fuzzy $QS$-subalgebra of $Y$ with the membership function $\mu_G$. Then the inverse image $f^{-1}[G]$ of $G$ is an i-v fuzzy $QS$-subalgebra of $X$.

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy $QS$-subalgebra of $Y$, by Theorem 3.2, we get that $\mu_B^L$ and $\mu_B^U$ are fuzzy $QS$-subalgebras of $Y$. By Proposition 2.2, $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy $QS$-subalgebras of $X$. By the above lemma and Theorem 3.2, we can conclude that $f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]$ is an i-v fuzzy $QS$-subalgebra of $X$. □

Definition 3.4. [2] Let $f$ be a mapping from the set $X$ into a set $Y$, and $A$ be an i-v fuzzy set in $X$ with membership function $\mu_A$. Then the image of $A$, denoted by $f[A]$, is the i-v fuzzy set in $Y$ with membership function defined by:

$$\mu_{f[A]}(y) = \begin{cases} \overset{\text{rsup}}{\sup} \{ \mu_A(z) \mid z \in f^{-1}(y) \} & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\ 0 & \text{otherwise} \end{cases}$$

in which $f^{-1}(y) = \{ x \mid f(x) = y \}$.

Theorem 3.9. Let $f$ be a $QS$-homomorphism from $X$ onto $Y$. If $A$ is an i-v fuzzy $QS$-subalgebra of $X$, then the image $f[A]$ of $A$ is an i-v fuzzy $QS$-subalgebra of $Y$.

Proof. Assume that $A$ is an i-v fuzzy $QS$-subalgebra of $X$, then $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy $QS$-subalgebra of $X$ if and only if $\mu_B^L$ and $\mu_B^U$ are fuzzy $QS$-subalgebras of $X$. By Proposition 2.3, $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy $QS$-subalgebras of $Y$. By Lemma 3.2, and Theorem 3.2, we can conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy $QS$-subalgebra of $Y$. □

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References
