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Note on Transformations of Posets with the Same Upper Bound Graph and Minimal Elements

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Abstract. Two posets with the same canonical poset and the same upper bound graph can be transformed into each other by a finite sequence of two kinds of transformations, called $x < y$-additions and $x < y$-deletions on minimal elements.

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1. Introduction

In this paper, we consider finite undirected simple graphs and finite posets. For a poset $P = (X, \leq)$, the *upper bound graph* (UB-graph) of $P = (X, \leq)$ is the graph $UB(P) = (X, E_{UB(P)})$, where $uv \in E_{UB(P)}$ if and only if $u \neq v$ and there exists $m \in X$ such that $u, v \leq_P m$. McMorris and Zaslavsky introduced this concept and gave a characterization of upper bound graphs [2].

Figure 1 shows two different posets which have the same upper bound graph. This example induces an interest in properties of posets with the same UB-graph.

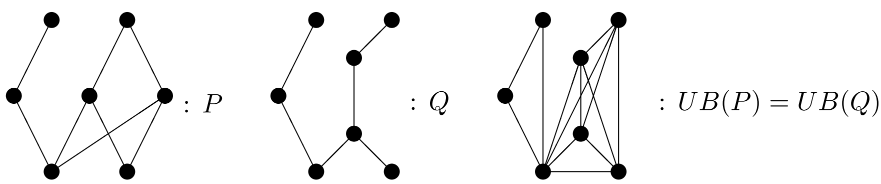

Figure 1. Posets P, Q and $UB(P) = UB(Q) = G$.

In [3] and [4] we deal with sequences of transformations that convert a poset to any other poset that has the same upper bound graph. In this paper we show that the transformations can be of a special kind involving minimal elements of the posets at each step.

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2. Transformations of posets

For a poset $P = (X, \leq)$ and $x \in X$, $L_P(x) = \{y \in X \ ; \ y < x\}$ and $U_P(x) = \{y \in X \ ; \ y > x\}$. Furthermore $V(P)$ is X, $\max(P)$ is the set of all maximal elements of P, $\min(P)$ is the set of all minimal elements of P. For a poset P and $x, y \in V(P)$, $x \parallel_P y$ shows that x is incomparable with y in P. For a poset P, the *canonical poset* of P is the poset can(P) on the set $V(P)$ in which $x \leq_{\mathrm{can(P)}} y$ if and only if (1) $y \in \max(P)$ and $x \leq_P y$, or (2) $x = y$.

A *clique* in the graph G is the vertex set of a maximal complete subgraph. In some cases, we consider that a clique is a maximal complete subgraph. We say a family $\mathcal{C}$ of complete subgraphs *edge covers* G if for each edge $uv \in E(G)$, there exists $C \in \mathcal{C}$ such that $u, v \in C$.

Theorem 2.1. [2] *Let G be a graph with n vertices. The graph G is a UB-graph if and only if there exists a family $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ of complete subgraphs of G such that*

(a) *$\mathcal{C}$ edge covers G,*
(b) *for each C_i, there exists a vertex $v_i \in C_i - (\bigcup_{j \neq i} C_j)$.*

Furthermore, such a family $\mathcal{C}$ must consist of cliques of G and is the only such family if G has no isolated vertices.

For a UB-graph G and an edge clique cover $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ satisfying the conditions of Theorem 2.1, a vertex subset $K_{UB}(G)$ that consists of one element of each set $C_i - (\bigcup_{j \neq i} C_j)$ is called a *kernel* of G. We know a fact that, given any $K_{UB}(G)$, there exists a poset P such that $G = UB(P)$ and $K_{UB}(G) = \max(P)$.

In the remainder of this paper, we consider a fixed labeled connected UB-graph G with a fixed kernel $K_{UB}(G)$.

We define $\mathcal{P}_{UB}(G) = \{P \ ; \ UB(P) = G, \max(P) = K_{UB}(G)\}$. Each poset P in $\mathcal{P}_{UB}(G)$ is identified with the set of comparable pairs in P. Thus $\mathcal{P}_{UB}(G)$ is a poset by set inclusion. The canonical poset can(G) of G is the canonical poset of any poset P in $\mathcal{P}_{UB}(G)$. By Theorem 2.1, the canonical poset is independent of the choice. For a UB-graph G, the canonical poset can(G) is a height 1 poset and $V(\mathrm{can(G)}) = \max(\mathrm{can(G)}) \cup \min(\mathrm{can(G)})$.

To consider some relations among posets of $\mathcal{P}_{UB}(G)$, we need some concepts as follows: For elements x and y in a poset P such that $y \notin \max(P)$ and x is covered by y, the poset $P^-_{x<y}$ is obtained from P by subtracting the relation $x \leq y$ from P, and we call this transformation the *$x < y$-deletion.* For an incomparable pair x and y in a poset P such that $y \notin \max(P)$, $U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$, the poset $P^+_{x<y}$ is obtained from P by adding the relation $x \leq y$ to P, and we call this transformation the *$x < y$-addition.* We obtain the following facts on these transformations.

Fact 1. For a poset P,

(1) P and $P^-_{x<y}$ have the same UB-graph,
(2) P and $P^+_{x<y}$ also have the same UB-graph, and
(3) $x < y$-additions and $x < y$-deletions are inverse transformations to each other.

By these facts, we obtained the following result.

Theorem 2.2. [3] *Let G be a UB-graph and P, Q be posets in $\mathcal{P}_{UB}(G)$. Then*

(1) *P can be transformed into Q by a sequence of $x < y$-deletions and $x < y$-additions.*

(2) *Every poset in $\mathcal{P}_{UB}(G)$ is obtained from* can(G) *by $x < y$-additions only.*

For an $x < y$-addition, we need to check two conditions: $U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$. If x is a minimal element of a poset P, $L_P(x) = \emptyset$ and we only check the condition $U_P(y) \subseteq U_P(x)$ for an $x < y$-addition on an incomparable pair x and y. The next result deals with $x < y$-additions on a minimal element x.

Theorem 2.3. *Let G be a UB-graph and P be a poset in $\mathcal{P}_{UB}(G)$. The poset P is obtained from* can(P) *by $x < y$-additions only, where x is a minimal element of the poset at each step.*

Proof. Since P is finite and every deletion reduces the number of comparable pairs in P, we obtain the following sequence of $x < y$-deletions, where x is a minimal element of the poset of each step:

$$P \xrightarrow{x<y-deletion} \dots \xrightarrow{x<y-deletion} \mathrm{can(P)} = \mathrm{can(G)}\ ,$$

In each step of this sequence of $x < y$-deletions, $U_{P^-_{x<y}}(x) \supseteq U_{P^-_{x<y}}(y)$, x is incomparable with y in $P^-_{x<y}$ and a minimal element of $P^-_{x<y}$, and $y \notin \max(P^-_{x<y})$. Thus the inverse operations of the above $x < y$-deletions are $x < y$-additions on x, y satisfying that $x \parallel_P y$, $U_P(x) \supseteq U_P(y)$, x is a minimal element of P and y is not a maximal element of P. So we obtain the following sequence of $x < y$-additions, where x is a minimal element of the poset at each step.

$$P \xleftarrow{x<y-addition} \dots \xleftarrow{x<y-addition} \mathrm{can(P)} = \mathrm{can(G)}.$$

□

From this result we obtain the next result.

Theorem 2.4. *Let G be a UB-graph and P, Q be posets in $\mathcal{P}_{UB}(G)$. The poset P can be transformed into Q by a sequence of $x < y$-deletions and $x < y$-additions, where x is a minimal element of the poset at each step and all the deletions can precede all the additions.*

Proof. As in the proof of Theorem 2.3, P can be reduced to can(G) by $x < y$-deletions and then can(G) can be enlarged to Q by $x < y$-additiions, where x is always a minimal element of the poset at each step. □

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