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On $\mathcal{K}$-Starcompact Spaces

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Abstract. A space X is $\mathcal{K}$-*starcompact* if for every open cover $\mathcal{U}$ of X, there exists a compact subset K of X such that $St(K,\mathcal{U}) = X$, where $St(K,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap K \neq \emptyset\}$. In this paper, we investigate the relations between $\mathcal{K}$-starcompact spaces and other related spaces. We also study topological properties of $\mathcal{K}$-starcompact spaces.

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1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [1] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$, and he proved that every countably compact space is starcompact. Conversely, van Douwen et al. [2] proved that every Hausdorff starcompact space is countably compact, but this does not hold for T_1-space (see [3, Example 2.5]. As generalizations of starcompactness, the following classes of spaces were given.

Definition 1.1. (4) A space X is $\mathcal{K}$-starcompact if for every open cover $\mathcal{U}$ of X, there exists a compact subset K of X such that $St(K,\mathcal{U}) = X$.

Definition 1.2. (5) A space X is $1\frac{1}{2}$-starcompact if for every open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U}) = X$.

In, a $1\frac{1}{2}$-starcompact space is called 1-starcompact. From the above definitions, it is not difficult to see that every starcompact space is $\mathcal{K}$-starcompact and every $\mathcal{K}$-starcompact space is $1\frac{1}{2}$-starcompact, but the converses do not hold (see Examples 2.1, 2.2 and 2.3 below).

Throughout the paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of continuum. As usual, a

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cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, All unexplained concepts or symbols are standard as in Engelking [6].

2. $\mathcal{K}$-starcompact spaces and related spaces

Fleischman [1] proved that every countably compact space is starcompact. By the above definitions, it is clear that every starcompact space is $\mathcal{K}$-starcompact and every $\mathcal{K}$-starcompact space is $1\frac{1}{2}$-starcompact. In [1], van Douwen et al. have proved that every $1\frac{1}{2}$-starcompact space is pseudocompact. Since every normal pseudocompact space is countably compact, thus we have the following theorem.

Theorem 2.1. *Let X be a normal space. Then, the following conditions are equivalent:*

(a) *X is countably compact;*
(b) *X is starcompact;*
(c) *X is $\mathcal{K}$-starcompact;*
(d) *X is $1\frac{1}{2}$-starcompact;*
(e) *X is pseudocompact.*

In the following, we show that Theorem 2.1 is not true for the classes of T_1 or Tychonoff spaces. van Douwen et al. [2] have proved that every starcompact Hausdorff space is countably compact, but this does not hold for T_1-spaces (see [3, Example 2.5]). For a Tychonoff space X, the symbol $\beta(X)$ means the Čech-Stone compactification of X.

Example 2.1. There exists a Tychonoff $\mathcal{K}$-starcompact space X which is not starcompact.

Proof. Let

$$X = (\beta(\omega) \times (\omega_1 + 1)) \setminus ((\beta(\omega) \setminus \omega) \times \{\omega_1\}).$$

To show that X is $\mathcal{K}$-starcompact. Let $\mathcal{U}$ be an open cover of X. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open subsets of X. For each $n \in \omega$, there exist an $\alpha_n < \omega_1$ and $U_n \in \mathcal{U}$ such that

$$\{n\} \times (\alpha_n, \omega_1] \subseteq U_n.$$

Let $\alpha = \sup\{\alpha_n : n \in \omega\}$. Then, $\alpha < \omega_1$. If we put $K_1 = \beta(\omega) \times \{\alpha + 1\}$, then K_1 is a compact subset of X and $\omega \times \{\omega_1\} \subseteq St(K_1, \mathcal{U})$, since $\langle n, \omega_1 \rangle \in U_n \in \mathcal{U}$ and $U_n \cap K_1 \neq \emptyset$, for each $n \in \omega$. On the other hand, since $\beta(\omega) \times \omega_1$ is countably compact, there is a finite subset F of $\beta(\omega) \times \omega_1$ such that

$$\beta(\omega) \times \omega_1 \subseteq St(F, \mathcal{U}).$$

If we put $K = K_1 \cup F$, then K is a compact subset of X and $X = St(K, \mathcal{U})$.This shows that X is $\mathcal{K}$-starcompact. ∎

However, X is not countably compact, since $\{\langle n, \omega_1 \rangle : n \in \omega\}$ is a closed discrete infinite subset of X. Thus, X is not starcompact, since every Hausdorff starcompact is countably compact.

Remark 2.1. Example 2.1 shows that the closed subset $\{\langle n, \omega_1\rangle : n \in \omega\}$ of a Tychonoff $\mathcal{K}$-starcompact space X is not $\mathcal{K}$-starcompact, since it is a closed discrete infinite subset of X.

Example 2.2. There exists a $1\frac{1}{2}$-starcompact T_1-space X which is not $\mathcal{K}$-starcompact.

Proof. Let $X = \omega_1 \cup A$, where $A = \{a_\alpha \ \alpha \in \omega_1\}$ is a set of cardinality ω_1. We topologize X as follows: ω_1 has the usual order topology and is an open subspace of X; a basic neighborhood of a point $a_\alpha \in A$ takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

Then, X is a T_1-space. To show that X is $1\frac{1}{2}$-starcompact, let $\mathcal{U}$ be an open cover of X. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open subsets of X. Thus, it is suffices to show that there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\cup\mathcal{V}, \mathcal{U}) = X$. Since ω_1 is countably compact, then it is $1\frac{1}{2}$-starcompact. Hence there is a finite subset $\mathcal{V}_1$ of $\mathcal{U}$ such that $\omega_1 \subseteq St(\cup\mathcal{V}_1, \mathcal{U})$. On the other hand, if we pick $\alpha_0 < \omega_1$, then there exists a $U_{\alpha_0} \in \mathcal{U}$ such that $a_{\alpha_0} \in U_{\alpha_0}$. For each $\alpha < \omega_1$, there is $U_\alpha \in \mathcal{U}$ such that $a_\alpha \in U_\alpha$., Hence we have $U_\alpha \cap U_{\alpha_0} \neq \emptyset$, by the definition of the topology of X. Therefore $A \subseteq St(U_{\alpha_0}, \mathcal{U})$. If we put $\mathcal{V} = \mathcal{V}_1 \cup \{U_{\alpha_0}\}$, then $\mathcal{V}$ is a finite subset of $\mathcal{U}$ and $X = St(\cup\mathcal{V}, \mathcal{U})$.

Next, we show that X is not $\mathcal{K}$-starcompact. Let us consider the open cover

$$\mathcal{V} = \{\omega_1\} \cup \{O_\alpha(a_\alpha) : \alpha < \omega_1\}.$$

Let K be a compact subset of X. Since A is discrete closed in X, there exists an $\alpha_1 < \omega_1$ such that

$$K \cap \{a_\alpha : \alpha > \alpha_1\} = \emptyset;$$

On the other hand, since ω_1 is a countably compact space, there exists $\alpha_2 < \omega_1$ such that $K \cap (\alpha_2, \omega_1) = \emptyset$, since $K \cap \omega_1$ is compact in ω_1. Choose $\beta > \max\{\alpha_1, \alpha_2\}$. Then $a_\beta \notin St(K, \mathcal{V})$, since $O_\beta(a_\beta)$ is the unique element of $\mathcal{V}$ containing a_β and $O_\beta \cap K = \emptyset$. This shows that X is not $\mathcal{K}$-starcompact. ∎

Question 1. Is there a $1\frac{1}{2}$-starcompact Hausdorff (or Tychonoff) space which is not $\mathcal{K}$-starcompact?

Example 2.3. [2, Example 2.2.5] There exists a pseudocompact Tychonoff space which is not $1\frac{1}{2}$-starcompact.

3. Topological properties of $\mathcal{K}$-starcompact spaces

In this section, we study topological properties of $\mathcal{K}$-starcompact spaces. Example 2.2 shows that a closed subset of a $\mathcal{K}$-starcompact space X need not be $\mathcal{K}$-starcompact. Now, we give an example showing that a regular closed subset of a Tychonoff $\mathcal{K}$-starcompact space need not be $\mathcal{K}$-starcompact. Here, a subset A of a space X is said to be regular closed in X if $A = \mathrm{cl}_X \,\mathrm{int}_X A$.

Example 3.1. There exists a $\mathcal{K}$-starcompact Tychonoff space X having a regular-closed subset which is not $\mathcal{K}$-starcompact.

Proof. Let D be a discrete space of cardinality $\mathfrak{c}$. Let

$$X = (\beta(D) \times (\mathfrak{c}^+ + 1)) \setminus ((\beta(D) \setminus D) \times \{\mathfrak{c}^+\}).$$

To show that X is $\mathcal{K}$-starcompact. For this end, let $\mathcal{U}$ be an open cover of X. Since $\beta(D) \times \mathfrak{c}^+$ is countably compact, then it is star-compact. Hence there exists a finite subset F of $\beta(D) \times \mathfrak{c}^+$ such that

$$\beta(D) \times \mathfrak{c}^+ \subseteq St(F, \mathcal{U}).$$

It remains to find a compact subset K_1 such that $D \times \{\mathfrak{c}^+\} \subseteq St(K_1, \mathcal{U})$. For each $d \in D$, there exists a $\alpha_d < \mathfrak{c}^+$ such that $\{d\} \times (\alpha_d, \mathfrak{c}^+]$ is included in some member of $\mathcal{U}$. Let $\alpha_0 = \sup\{\alpha_d : d \in D\}$. Then, $\alpha_0 < \mathfrak{c}^+$, since $|D| = \mathfrak{c}$. Thus, if we put $K_1 = \beta(D) \times \{\alpha_0 + 1\}$, then K_1 is a compact subset of X and $D \times \{\mathfrak{c}^+\} \subseteq St(K_1, \mathcal{U})$. If we put $K = F \cup K_1$, then K is a compact subset of X and $X = St(K, \mathcal{U})$, which completes the proof. Let $\Psi = \omega \cup \mathcal{R}$ be the Isbell-Mrŏwka space, where $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$ (see Mrówka [7]). Then, the space Ψ is not $1\frac{1}{2}$-starcompact [2, Example 2.25], and hence, it is not $\mathcal{K}$-starcompact, since every $\mathcal{K}$-starcompact space is $1\frac{1}{2}$-starcompact.

Assume $X \cap \Psi = \emptyset$. Define a bijection $f : D \times \{\mathfrak{c}^+\} \to \mathcal{R}$. Let Y be the quotient space obtained from the topological sum $X \oplus \Psi$ by identifying p with $f(p)$ for every $p \in D \times \{\mathfrak{c}^+\}$ and let $\varphi : X \oplus \Psi \to Y$ be the quotient map. Then, $\varphi(\Psi)$ is a regular-closed subspace of Y which is not $\mathcal{K}$-starcompact, since $\varphi(\Psi)$ is homemorphic to Ψ.

Next, we show that Y is $\mathcal{K}$-starcompact. For this end, let $\mathcal{U}$ be an open cover of Y. Then, there exists a compact subset F_1 of $\varphi(X)$ such that

$$\varphi(X) \subseteq St(F_1, \mathcal{U}),$$

since $\varphi(X)$ is homemorphic to X. Since every infinite subset of ω has a limit point in Ψ, the set $F_2 = Y \setminus St(F_1, \mathcal{U})$ is finite. Consequently, if we put $K = F_1 \cup F_2$, then K is a compact subset of Y and $Y = St(K, \mathcal{U})$. Therefore, Y is $\mathcal{K}$-starcompact completing the proof. ∎

Concerning the image and preimage of a $\mathcal{K}$-starcompact space under a continuous map, Ikenaga and Tani [4] have proved the following two theorems.

Theorem 3.1. [4] *Let $f : X \to Y$ be a continous map from a $\mathcal{K}$-starcompact space X onto a space Y. Then, Y is a $\mathcal{K}$-starcompact space.*

Theorem 3.2. [4] *Let $f : X \to Y$ be an open perfect continuous map from a space X onto a $\mathcal{K}$-starcompact space. Then, X is a $\mathcal{K}$-starcompact space.*

The following example shows that the condition of open perfect can not be replaced by perfect in the Theorem 3.2. We use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Example 3.2. There exists a perfect onto map $f : X \to Y$ such that Y is a $\mathcal{K}$-starcompact space, but X is not $\mathcal{K}$-starcompact.

Proof. Let Y be the space X in the proof of Example 3.1. Then Y is $\mathcal{K}$-starcompact and has the infinite discrete closed subset $F = D \times \{\mathfrak{c}^+\}$. Let X be the Alexandroff duplicate $A(Y)$ of Y. Then, X is not $\mathcal{K}$-starcompact, since $F \times \{1\}$ is an infinite discrete, open and closed set in X. Let $f : X \to Y$ be the natural map. Then, f is a perfect map, This completes the proof. ∎

By Theorem 3.2, we have the following corollary.

Corollary 3.1. *The product of a $\mathcal{K}$-starcompact space and a compact space is $\mathcal{K}$-starcompact.*

However, the product of two $\mathcal{K}$-starcompact spaces need not be $\mathcal{K}$-starcompact. In fact, the product of two countably compact spaces is not necessarily $\mathcal{K}$-star-ompact.

Example 3.3. There exist two countably compact spaces X and Y such that $X \times Y$ is not $\mathcal{K}$-starcompact.

Proof. We define $X = \cup_{\alpha<\omega_1} E_\alpha$, $Y = \cup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(\omega)$ which are defined inductively by the following conditions:

(1) $E_\alpha \cap F_\beta = \omega$ if $\alpha \neq \beta$;
(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
(3) every infinite subset of E_α (resp. F_α) has an accumullation point in $E_{\alpha+1}$(resp. $F_{\alpha+1}$).

Those sets E_α and F_α are well-defined since every infinite closed set in $\beta(\omega)$ has the cardinality $2^{\mathfrak{c}}$. Then, $X \times Y$ is not $\mathcal{K}$-starcompact, because the diagonal $\{\langle n, n\rangle : n \in \omega\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality ω and $\mathcal{K}$-starcompactness is preserved by open and closed subset. ∎

We end this section by the following theorem. For a space X let $l(X)$ denote the *Lindelöf number* of the space X; that is, the smallest cardinal number κ such that every open cover of X has an open refinement $\mathcal{V}$ with $|\mathcal{V}| \leq \kappa$.

Theorem 3.3. *Every Tychonoff space can be embedded in a $\mathcal{K}$-starcompact Tychonoff space as a closed subspace.*

Proof. Let X be a Tychonoff space. If we put

$$Z = (\beta(X) \times (\tau^+ + 1)) \setminus (\beta(X) \times \{\tau^+\}),$$

where τ is a regular cardinal and $\tau > l(X)$, then $\overline{X} = X \times \{\tau^+\}$ is a closed subset of Z, which is homeomrphic to X. To show that Z is $\mathcal{K}$-starcompact. Let $\mathcal{U}$ be an open cover of Z. Since $\beta(X) \times \tau^+$ is countably compact, there is a finite subset $F_1 \subseteq$ such that

$$\beta(X) \times \tau^+ \subseteq St(F_1, \mathcal{U}).$$

It remains to find a compact subset $F_2 \subseteq Z$ such that $\overline{X} \subseteq St(F_2, \mathcal{U})$. Denote by $\mathcal{V}$ the family of all sets of the from $V = (W(V) \times (\alpha(V), \tau^+]) \cap Z$ (where $\alpha(V) < \tau^+$ and $W(V)$ is an open set in $\beta(X)$) such that $V \subseteq U(V)$ for some $U(V) \in \mathcal{U}$. Then $\mathcal{V}$ is an open cover of $\overline{X}$. There exists a subcover $\mathcal{V}_0 \subseteq \mathcal{V}$ of cardinality $\leq l(X)$. If we put

$$\alpha^* = \sup\{\alpha(V) : V \in \mathcal{V}_0\} + 1,$$

then $F_2 = \beta(X) \times \{\alpha^*\}$ is a compact subspace of Z. Since every element of $\mathcal{V}_0$ intersects F_2, we have

$$\overline{X} \subseteq St(F_2, \mathcal{V}_0) \subseteq St(F_2, \mathcal{U}).$$

Set $K = F_1 \cup F_2$; then K is a compact subset of Z and $Z \subseteq St(K, \mathcal{U})$, which completes the proof. ∎

Question 2. Can a Tychonoff space be embedded in a $\mathcal{K}$-starcompact Tychonoff space as a G_δ-closed subspace?

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