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On Automatic Continuity of 3-Homomorphisms on Banach Algebras

[1]Janko Bračič and [2]Mohammad Sal Moslehian
[1]Department of Mathematics, Univ. of Ljubljana, Jadranska ul. 19, 1111 Ljubljana, Slovenia
[2]Department of Mathematics, Ferdowsi University, P. O. Box 1159, Mashhad 91775, Iran
Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi Univ., Iran
Banach Mathematical Research Group (BMRG), Mashhad, Iran.
[1]janko.bracic@fmf.uni-lj.si, [2]moslehian@ferdowsi.um.ac.ir

Abstract. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between (Banach) algebras is called 3-homomorphism if $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$ for each $a, b, c \in \mathcal{A}$. We investigate 3-homomorphisms on Banach algebras with bounded approximate identities and establish in two ways (for unital and non-unital cases) that every involution preserving homomorphism between C^*-algebras is norm decreasing.

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1. Introduction

In Hejazian et al. [3] the notion of 3-homomorphism between (Banach) algebras was introduced and some of their significant properties were investigated. A linear mapping $\varphi : \mathcal{A} \to \mathcal{B}$ between (Banach) algebras is called 3-homomorphism if $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$ for each $a, b, c \in \mathcal{A}$. Obviously, each homomorphism is a 3-homo-morphism. One can verify that if $\mathcal{A}$ is unital and φ is a 3-homomorphism then $\psi(a) = \varphi(1)\varphi(a)$ is a homomorphism.

There might exist 3-homomorphisms that are not homomorphisms. For instance, if $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $\varphi := -\psi$ is a 3-homomorphism which is not a homomorphism. As another example, assume that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras such that $\mathcal{A}^3 = \{0\}$ and $\mathcal{B}^3 = \{0\}$. Then each linear mapping between $\mathcal{A}$ and $\mathcal{B}$ is trivially a 3-homomorphism. One can find some non-trivial examples in [3].

In this paper, we investigate 3-homomorphisms on Banach algebras with bounded approximate identities. In particular we establish in two ways (for unital and non-unital cases) that every involution preserving homomorphism between C^*-algebras is norm decreasing.

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Throughout the paper, we denote by T^* the dual of a linear mapping T and by $\mathcal{A}^*$ the dual of a Banach algebra $\mathcal{A}$. We also denote the involution on a C^*-algebra by $\#$.

2. 3-Homomorphisms on C^*-algebras

Suppose that φ is a non-zero 3-homomorphism from a unital algebra $\mathcal{A}$ to $\mathbb{C}$. Then $\varphi(1) = 1$ or -1. Hence either φ or $-\varphi$ is a character on $\mathcal{A}$. If $\mathcal{A}$ is a Banach algebra, then φ is automatically continuous [1, Theorem 2.1.29]. It may happen, however, that a 3-homomorphism is not continuous. For instance, let $\mathcal{A}$ be the algebra of all 3×3 matrices having 0 on and below the diagonal and $\mathcal{B}$ be the algebra of all $\mathcal{A}$-valued continuous functions from $[0, 1]$ into $\mathcal{A}$ with sup norm. Then $\mathcal{B}$ is an infinite dimensional Banach algebra in which the product of any three elements is 0. Since $\mathcal{B}$ is infinite dimensional there are linear discontinuous maps and so discontinuous 3-homomorphisms from $\mathcal{B}$ into itself.

Using extreme points of the closed unit ball of a W^*-algebra $\mathcal{A}$ authors of [3] showed that if $\mathcal{B}$ is a C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a weakly-norm continuous involution preserving 3-homomorphism, then $\|\varphi\| \leq 1$. Then they asked whether every preserving involution 3-homomorphism between C^*-algebras is continuous. We shall establish this by applying the strategy used by Harris [2, Proposition 3.4] where it is shown that each J^*-homomorphism between J^*-algebras is norm decreasing.

Theorem 2.1. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an involution preserving 3-homomorphism between unital C^*-algebras. Then $\|\varphi\| \leq 1$.*

Proof. Let $a \in \mathcal{A}$, let $\lambda > 0$ and let $\lambda \notin \sigma(a^\# a)$. Then $\lambda 1_{\mathcal{A}} - a^\# a$ has an inverse b. Then we have the following sequence of deductions:

$$\begin{aligned}
ab(\lambda 1_{\mathcal{A}} - a^\# a) &= a \\
(\lambda b - ba^\# a)a(\lambda b - ba^\# a) &= a \\
\varphi(\lambda b - ba^\# a)\varphi(a)\big(\lambda\varphi(b) - \varphi(b)\varphi(a^\#)\varphi(a)\big) &= \varphi(a) \\
\varphi(a^\#)\left(\varphi(\lambda b - ba^\# a)\varphi(a)\varphi(b)\right)\left(\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a)\right) &= \varphi(a^\#)\varphi(a) \\
\varphi(a^\#)\varphi(ab)\big(\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a)\big) &= \varphi(a^\#)\varphi(a) \\
\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(ab)\big(\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a)\big) &= \lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a) \\
1_{\mathcal{B}} &= \frac{1}{\lambda}\big(1_{\mathcal{B}} + \varphi(a^\#)\varphi(ab)\big) \\
&\quad \times \big(\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a)\big).
\end{aligned}$$

Since $\lambda 1_{\mathcal{B}} - \varphi(a^\#)\varphi(a)$ is self-adjoint, it follows that $\lambda 1_{\mathcal{B}} - \varphi(a)^\# \varphi(a)$ is invertible and so $\lambda \notin \sigma\big(\varphi(a)^\# \varphi(a)\big)$. Hence

$$\sigma\big(\varphi(a)^\# \varphi(a)\big) \subseteq \sigma(a^\# a) \cup \{0\}.$$

Since the norm of a normal element c of a C^*-algebra is equal to its spectral radius $r(c)$ (see [1, Theorem 3.2.3]), we have

$$\|\varphi(a)\|^2 = \|\varphi(a^\# a)\| = r\big(\varphi(a)^\# \varphi(a)\big) \leq r(a^\# a) = \|a^\# a\| = \|a\|^2.$$

So φ is norm decreasing. ∎

3. 3-homomorphisms from algebras with bounded approximate identity

Let $\mathcal{A}$ be a Banach algebra. Let $a \in \mathcal{A}$ and $\xi \in \mathcal{A}^*$ be arbitrary, then $a.\xi \in \mathcal{A}^*$ and $\xi.a \in \mathcal{A}^*$ are defined by $\langle a.\xi, x\rangle = \langle \xi, xa\rangle, x \in \mathcal{A}$ and $\langle \xi.a, x\rangle = \langle \xi, ax\rangle, x \in \mathcal{A}$.
For $\xi \in \mathcal{A}^*$ and $F \in \mathcal{A}^{**}$, functionals $F.\xi \in \mathcal{A}^*$ and $\xi.F \in \mathcal{A}^*$ are given by $\langle F.\xi, x\rangle = \langle F, \xi.x\rangle, x \in \mathcal{A}$ and $\langle \xi.F, x\rangle = \langle F, x.\xi\rangle, x \in \mathcal{A}$. Recall that the first Arens product of $F, G \in \mathcal{A}^{**}$ is $F \triangleleft G \in \mathcal{A}^{**}$, where

$$\langle F \triangleleft G, \xi\rangle = \langle F, G.\xi\rangle \qquad (\xi \in \mathcal{A}^*).$$

Similarly, the second Arens product of $F, G \in \mathcal{A}^{**}$ is $F \triangleright G \in \mathcal{A}^{**}$, where

$$\langle F \triangleright G, \xi\rangle = \langle G, \xi.F\rangle \qquad (\xi \in \mathcal{A}^*).$$

It is well known that $(\mathcal{A}^{**}, \triangleleft)$ and $(\mathcal{A}^{**}, \triangleright)$ are Banach algebras and that $\mathcal{A}$ is a closed subalgebra of each of them. The algebra $\mathcal{A}$ is Arens regular if the products $\triangleleft$ and $\triangleright$ coincide on $\mathcal{A}^{**}$.

A left approximate identity for $\mathcal{A}$ is a net $(e_\alpha) \subset \mathcal{A}$ such that $\lim_\alpha e_\alpha a = a$, for each $a \in \mathcal{A}$. If there exists $m > 0$ such that $\sup_\alpha \|e_\alpha\| \le m$, the left approximate identity (e_α) is said to be bounded by m. The definitions of a right approximate identity and a bounded right approximate identity are similar. Recall from [1, Proposition 2.9.16] that $\mathcal{A}$ has a left approximate identity bounded by m if and only if $(\mathcal{A}^{**}, \triangleright)$ has a left identity of norm at most m. Similarly, $\mathcal{A}$ has a right approximate identity bounded by m if and only if $(\mathcal{A}^{**}, \triangleleft)$ has a right identity of norm at most m.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a 3-homomorphism. Then, for arbitrary $a, b \in \mathcal{A}$, $\xi \in \mathcal{B}^*$, and $F, G \in \mathcal{A}^{**}$, the following hold:*

(i) $\varphi^*(\varphi(a)\varphi(b).\xi) = ab.\varphi^*(\xi)$;
(ii) $\varphi^*(\xi.\varphi(a)\varphi(b)) = \varphi^*(\xi).ab$;
(iii) $\varphi^*(\varphi(a).\xi.\varphi^{**}(F)) = a.\varphi^*(\xi).F$;
(iv) $\varphi^*(\varphi^{**}(F).\xi.\varphi(a)) = F.\varphi^*(\xi).a$;
(v) $\varphi^*(\xi.\varphi^{**}(F) \triangleright \varphi^{**}(G)) = \varphi^*(\xi).F \triangleright G$;
(vi) $\varphi^*(\varphi^{**}(F) \triangleleft \varphi^{**}(G).\xi) = F \triangleleft G.\varphi^*(\xi)$.

Proof. (i) Let $a, b \in \mathcal{A}$, and $\xi \in \mathcal{B}^*$ be arbitrary. Then, for each $x \in \mathcal{A}$,

$$\begin{aligned}\langle \varphi^*(\varphi(a)\varphi(b).\xi), x\rangle &= \langle \varphi(a)\varphi(b).\xi, \varphi(x)\rangle \\ &= \langle \xi, \varphi(x)\varphi(a)\varphi(b)\rangle \\ &= \langle \xi, \varphi(xab)\rangle \\ &= \langle \varphi^*(\xi), xab\rangle \\ &= \langle ab.\varphi^*(\xi), x\rangle.\end{aligned}$$

(iii) Let $a \in \mathcal{A}$, $\xi \in \mathcal{B}^*$, and $F \in \mathcal{A}^{**}$ be arbitrary. Then, for each $x \in \mathcal{A}$,

$$\begin{aligned}\langle \varphi^*(\varphi(a).\xi.\varphi^{**}(F)), x\rangle &= \langle \varphi(a).\xi.\varphi^{**}(F), \varphi(x)\rangle \\ &= \langle \varphi^{**}(F), \varphi(x)\varphi(a).\xi\rangle \\ &= \langle F, \varphi^*(\varphi(x)\varphi(a).\xi)\rangle \\ &= \langle F, xa.\varphi^*(\xi)\rangle \\ &= \langle a.\varphi^*(\xi).F, x\rangle.\end{aligned}$$

(v) Let $\xi \in \mathcal{B}^*$, and $F, G \in \mathcal{A}^{**}$ be arbitrary. Then, for each $x \in \mathcal{A}$,

$$\begin{aligned} \langle \varphi^*(\xi.\varphi^{**}(F) \triangleright \varphi^{**}(G)), x\rangle &= \langle(\xi.\varphi^{**}(F) \triangleright \varphi^{**}(G)), \varphi(x)\rangle \\ &= \langle \varphi^{**}(G), \varphi(x).\xi.\varphi^{**}(F)\rangle \\ &= \langle G, \varphi^*(\varphi(x).\xi.\varphi^{**}(F))\rangle \\ &= \langle G, x.\varphi^*(\xi).F\rangle \\ &= \langle \varphi^*(\xi).F \triangleright G, x\rangle. \end{aligned}$$

The proofs of (ii), (iv), and (vi) are similar. ∎

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a 3-homomorphism, then φ^{**} is a 3-homomorphism from $(\mathcal{A}^{**}, \triangleleft)$ to $(\mathcal{B}^{**}, \triangleleft)$ and from $(\mathcal{A}^{**}, \triangleright)$ to $(\mathcal{B}^{**}, \triangleright)$.*

Proof. Let $F, G, H \in \mathcal{A}^{**}$ be arbitrary. Then, for each $\xi \in \mathcal{B}^*$,

$$\begin{aligned} \langle \varphi^{**}(F \triangleleft G \triangleleft H), \xi\rangle &= \langle F \triangleleft G \triangleleft H, \varphi^*(\xi)\rangle \\ &= \langle F, G \triangleleft H.\varphi^*(\xi)\rangle \\ &= \langle F, \varphi^*(\varphi^{**}(G) \triangleleft \varphi^{**}(H).\xi)\rangle \\ &= \langle \varphi^{**}(F), \varphi^{**}(G) \triangleleft (\varphi^{**}(H).\xi)\rangle \\ &= \langle \varphi^{**}(F) \triangleleft \varphi^{**}(G), \varphi^{**}(H).\xi\rangle \\ &= \langle \varphi^{**}(F) \triangleleft \varphi^{**}(G) \triangleleft \varphi^{**}(H), \xi\rangle, \end{aligned}$$

where the second equality holds by Lemma 3.1 (vi).
The proof of the second assertion is similar. ∎

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a 3-homomorphism. If $\mathcal{A}$ has a bounded left approximate identity, then*

$$\psi(F) := \varphi^{**}(L) \triangleright \varphi^{**}(F) \qquad (F \in \mathcal{A}^{**})$$

defines a homomorphism from $(\mathcal{A}^{**}, \triangleright)$ to $(\mathcal{B}^{**}, \triangleright)$, where L is a left identity in $\mathcal{A}^{**}$. Moreover,

$$\varphi(a) = \varphi^{**}(L) \triangleright \psi(a) \qquad (a \in \mathcal{A}).$$

Similarly, if $\mathcal{A}$ has a bounded right approximate identity, then

$$\psi(F) := \varphi^{**}(F) \triangleleft \varphi^{**}(R) \qquad (F \in \mathcal{A}^{**})$$

defines a homomorphism from $(\mathcal{A}^{**}, \triangleleft)$ to $(\mathcal{B}^{**}, \triangleleft)$, where R is a left identity in $\mathcal{A}^{**}$. Moreover,

$$\varphi(a) = \psi(a) \triangleleft \varphi^{**}(R) \qquad (a \in \mathcal{A}).$$

Proof. We shall prove only the first part of the theorem. Since φ is a 3-homomorphism the mapping φ^{**} is a 3-homomorphism from $(\mathcal{A}^{**}, \triangleright)$ to $(\mathcal{B}^{**}, \triangleright)$, by Proposition 3.1. By [1, Proposition 2.9.16], there exists a left identity L in $(\mathcal{A}^{**}, \triangleright)$. Thus

$$\psi(F) := \varphi^{**}(L) \triangleright \varphi^{**}(F) \qquad (F \in \mathcal{A}^{**})$$

defines a homomorphism from $(\mathcal{A}^{**}, \triangleright)$ to $(\mathcal{B}^{**}, \triangleright)$ and

$$\varphi^{**}(F) = \varphi^{**}(L) \triangleright \psi(F) \qquad (F \in \mathcal{A}^{**}).$$

Since the restriction of φ^{**} to $\mathcal{A}$ is φ, we have

$$\varphi(a) = \varphi^{**}(L) \triangleright \psi(a) \quad (a \in \mathcal{A}).$$

∎

We now give another proof of Theorem 2.1:

Corollary 3.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Then each involution preserving 3-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is norm decreasing.*

Proof. Each C^*-algebra has approximate identity of bound 1 (cf. [1, Theorem 3.2.21]). By [1, Corollary 3.2.37], $\mathcal{A}$ and $\mathcal{B}$ are Arens regular and $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ are von Neumann algebras equipped with the involution that are defined in the following way (see [1, p. 349]). A linear involution on $\mathcal{A}^*$ is defined by

$$\langle \xi^{\#}, a\rangle = \overline{\langle \xi, a^{\#}\rangle} \quad (a \in \mathcal{A}, \xi \in \mathcal{A}^*),$$

and an involution on $\mathcal{A}^{**}$ is given by

$$\langle F^{\#}, \xi\rangle = \overline{\langle F, \xi^{\#}\rangle} \quad (F \in \mathcal{A}^{**}, \xi \in \mathcal{A}^*).$$

The involution on $\mathcal{B}^{**}$ is defined similarly. The algebra $\mathcal{A}^{**}$ is unital, with identity E of norm $\|E\| = 1$.

It is easy to check that φ^{**} is an involution preserving 3-homomorphism. Namely, for each $a \in \mathcal{A}$ and each $\xi \in \mathcal{B}^*$, we have

$$\langle \varphi^*(\xi^{\#}), a\rangle = \overline{\langle \xi, \varphi(a)^{\#}\rangle} = \overline{\langle \xi, \varphi(a^{\#})\rangle} = \langle \varphi^*(\xi)^{\#}, a\rangle.$$

It follows, for each $F \in \mathcal{A}^{**}$ and each $\xi \in \mathcal{B}^*$, that

$$\langle \varphi^{**}(F^{\#}), \xi\rangle = \overline{\langle F, \varphi^*(\xi)^{\#}\rangle} = \overline{\langle F, \varphi^*(\xi^{\#})\rangle} = \langle \varphi^{**}(F)^{\#}, \xi\rangle.$$

Since

$$\begin{aligned} \varphi^{**}(E)\varphi^{**}(F) &= \varphi^{**}(E)\varphi^{**}(FE^2) \\ &= \varphi^{**}(E)\varphi^{**}(F)\varphi^{**}(E)^2 \\ &= \varphi^{**}(EFE)\varphi^{**}(E) \\ &= \varphi^{**}(F)\varphi^{**}(E), \end{aligned}$$

for each $F \in \mathcal{A}^{**}$, we conclude that $\psi(F) = \varphi^{**}(E)\varphi^{**}(F) \quad (F \in \mathcal{A}^{**})$ defines an involution preserving homomorphism. Indeed, for each $F \in \mathcal{A}^{**}$, we have

$$\psi(F^{\#}) = \varphi^{**}(E)\varphi^{**}(F^{\#}) = (\varphi^{**}(F)\varphi^{**}(E)^{\#})^{\#} = \psi(F)^{\#}.$$

By [1, Corollary 3.2.4], each involution preserving homomorphism between C^*-algebras is norm-decreasing. Thus

$$\|\varphi^{**}(F)\| = \|\varphi^{**}(E)\psi(F)\| \leq \|\varphi^{**}(E)\|\|F\| \quad (F \in \mathcal{A}^{**}).$$

Since

$$\psi(E) = \varphi^{**}(E)^2 = \varphi^{**}(E)^{\#}\varphi^{**}(E)$$

we have

$$1 \geq \|\psi(E)\| = \|\varphi^{**}(E)^{\#}\varphi^{**}(E)\| = \|\varphi^{**}(E)\|^2,$$

which gives $\|\varphi^{**}(E)\| \leq 1$. Thus,

$$\|\varphi^{**}(F)\| \leq \|\varphi^{**}(E)\|\|F\| \leq \|F\| \quad (F \in \mathcal{A}^{**}).$$

∎

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