BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **30**(2) (2007), 99–110

On RW-Closed Sets in Topological Spaces

[1]S.S. BENCHALLI AND [2]R.S. WALI
Department of Mathematics, Karnatak University, Dharwad-580003, Karnataka, India
[1]benchalli_math@yahoo.com, [2]rswali@rediffmail.com

Abstract. In this paper, a new class of sets called regular w-closed (briefly rw-closed) sets in topological spaces is introduced and studied. A subset A of a topological space (X, τ) is called rw-closed if U contains closure of A whenever U contains A and U is regular semiopen in (X, τ). This new class of sets lies between the class of all w-closed sets and the class of all regular g-closed sets. Some of their properties are investigated.

2000 Mathematics Subject Classification: 54A05

Key words and phrases: Regular semiopen sets, rw-closed sets.

1. Introduction

Regular open sets and strong regular open sets have been introduced and investigated by Stone [32] and Tong [33] respectively. Levine [18,19], Biswas [5], Cameron [6], Sundaram and Sheik John [31], Bhattacharyya and Lahiri [4], Nagaveni [25], Pushpalatha [30], Gnanambal [15], Gnanambal and Balachandran [16], Palaniappan and Rao [28], Maki, Devi and Balachandran [21], and Park and Park [29] introduced and investigated semiopen sets, generalized closed sets, regular semiopen sets, weakly closed sets, semi-generalized closed sets, weakly generalized closed sets, strongly generalized closed sets, generalized pre-regular closed sets, regular generalized closed sets, generalized α-closed sets and α-generalized closed sets and mildly generalized closed sets respectively. We introduce a new class of sets called regular w-closed sets (rw-closed sets) which is properly placed in between the class of w-closed sets and the class of regular generalized closed sets.

Throughout this paper, space (X, τ) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X respectively.

Received: May 15, 2006; *Revised:* June 17, 2006.

2. Preliminaries

Definition 2.1. *A subset A of a topological space (X, τ) is called*

(i) *Regular open* [32] *if* $A = \mathrm{int}(\mathrm{cl}(A))$ *and regular closed* [32] *if* $A = \mathrm{cl}(\mathrm{int}(A))$.

(ii) *Pre-open* [24] *if* $A \subseteq \mathrm{int}(\mathrm{cl}(A))$ *and pre-closed* [24] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$.

(iii) *Semiopen* [18] *if* $A \subseteq \mathrm{cl}(\mathrm{int}(A))$ *and semiclosed* [7] *if* $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$.

(iv) *α-open* [27] *if* $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ *and α-closed* [23] *if* $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq A$.

(v) *Semi-preopen* [2] (= *β-open* [1]) *if* $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ *and semi-preclosed* [2] (= *β-closed* [1]) *if* $\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) \subseteq A$.

(vi) *θ-closed* [35] *if* $A = \mathrm{cl}_\theta(A)$, *where* $\mathrm{cl}_\theta(A) = \{x \in X : \mathrm{cl}(U) \cap A \neq \emptyset, U \in \tau$ *and* $x \in U\}$.

(vii) *δ-closed* [35] *if* $A = \mathrm{cl}_\delta(A)$ *where* $\mathrm{cl}_\delta(A) = \{x \in X : \mathrm{int}(\mathrm{cl}(U)) \cap A \neq \emptyset, U \in \tau$ *and* $x \in U\}$.

The intersection of all semiclosed (resp. semiopen) subsets of (X, τ) containing A is called the semi-closure (resp. semi-kernel) of A and is denoted by $\mathrm{scl}(A)$ (resp. $\mathrm{sker}(A)$). Also the intersection of all pre-closed (resp. semi-preclosed and α-closed) subsets of (X, τ) containing A is called the pre-closure (resp. semi-pre-closure and α-closure) of A and is denoted by $\mathrm{pcl}(A)$ (resp. $\mathrm{spcl}(A)$ and α-$\mathrm{cl}(A)$).

Definition 2.2. [11] *Let X be a topological space. The finite union of regular open sets in X is said to be π-open. The complement of a π-open set is said to be π-closed.*

Definition 2.3. [6] *A subset A of a space (X, τ) is called regular semiopen if there is a regular open set U such that $U \subset A \subset \mathrm{cl}(U)$. The family of all regular semiopen sets of X is denoted by* $\mathrm{RSO}(X)$.

Lemma 2.1. *Every regular semiopen set in (X, τ) is semiopen but not conversely.*

Proof. Follows from the definitions. ∎

Lemma 2.2. [14] *If A is regular semiopen in (X, τ), then $X \backslash A$ is also regular semiopen.*

Lemma 2.3. [14] *In a space (X, τ), the regular closed sets, regular open sets and clopen sets are regular semiopen.*

Definition 2.4. [8] *A subset A of a space (X, τ) is said to be semi-regular open if it is both semiopen and semiclosed.*

The family of all semi-regular open sets of X is denoted by $\mathrm{SR}(X)$. On other hand, Maio and Noiri [8] defined a subset A of X to be semi-regular open if $A = \mathrm{sint}(\mathrm{scl}(A))$. However, these three notions are equivalent, which is given in the following theorem.

Theorem 2.1. [8] *For a subset A of space X, the followings are equivalent:*

(i) $A \in \mathrm{SR}(X) (= \mathrm{RSO}(X))$,

(ii) $A = \mathrm{sint}(\mathrm{scl}(A))$,

(iii) *there exists a regular open set U of X such that* $U \subset A \subset \mathrm{cl}(U)$.

Definition 2.5. *A subset of a topological space (X,τ) is called*

(i) *Generalized closed (briefly g-closed)* [19] *if* $\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(ii) *Semi-generalized closed (briefly sg-closed)* [4] *if* $\mathrm{scl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semiopen in* X.

(iii) *Generalized semiclosed (briefly gs-closed)* [3] *if* $\mathrm{scl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(iv) *Generalized α-closed (briefly gα-closed)* [21] *if* $\alpha\text{-}\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is α-open in* X.

(v) *α-generalized closed (briefly αg-closed)* [22] *if* $\alpha\text{-}\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(vi) *Generalized semi-preclosed (briefly gsp-closed)* [9] *if* $\mathrm{spcl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(vii) *Regular generalized closed (briefly rg-closed)* [28] *if* $\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

(viii) *Generalized preclosed (briefly gp-closed)* [20] *if* $\mathrm{pcl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(ix) *Generalized pre regular closed (briefly gpr-closed)* [15] *if* $\mathrm{pcl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

(x) *θ-generalized closed (briefly θ-g-closed)* [12] *if* $\mathrm{cl}_\theta(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(xi) *δ-generalized closed (briefly δ-g-closed)* [10] *if* $\mathrm{cl}_\delta(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(xii) *Weakly generalized closed (briefly wg-closed)* [25] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(xiii) *Strongly generalized closed* [31] *(briefly g*-closed* [34]*) if* $\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is g-open in* X.

(xiv) *π-generalized closed (briefly πg-closed)* [11] *if* $\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is π-open in* X.

(xv) *Weakly closed (briefly w-closed)* [30] *if* $\mathrm{cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semiopen in* X.

(xvi) *Mildly generalized closed (briefly mildly g-closed)* [29] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is g-open in* X.

(xvii) *Semi weakly generalized closed (briefly swg-closed)* [25] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semiopen in* X.

(xviii) *Regular weakly generalized closed (briefly rwg-closed)* [25] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

The complements of the above mentioned closed sets are their respective open sets.

3. Rw-closed sets in topological spaces

Definition 3.1. *A subset A of a space (X,τ) is called regular w-closed (briefly rw-closed) if* $\mathrm{cl}(A) \subset U$ *whenever* $A \subset U$ *and* U *is regular semiopen in* (X,τ). *We denote the set of all rw-closed sets in* (X,τ) *by* $\mathrm{RWC}(X)$.

First we prove that the class of rw-closed sets properly lies between the class of w-closed sets and the class of regular generalized closed sets.

Theorem 3.1. *Every w-closed set in X is rw-closed in X but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular semiopen set is semiopen. ∎

The converse of the above theorem need not be true as seen from the following example.

Example 3.1. Let $X = \{a, b, c\}$ be with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the set $A = \{a, b\}$ is rw-closed, but not w-closed in X.

Theorem 3.2. *Every rw-closed set is regular generalized closed in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular open set is regular semiopen. ∎

The converse of the above theorem need not be true as seen from the following example.

Example 3.2. Let $X = \{a, b, c, d\}$ be with $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $A = \{c\}$ is regular generalized closed but not rw-closed in X.

Corollary 3.1. *Every closed set is rw-closed but not conversely.*

Proof. Follows from Sundaram and Sheik John [31] and Theorem 3.1. ∎

Corollary 3.2. *Every regular closed set is rw-closed but not conversely.*

Proof. Follows from Stone [32] and Corollary 3.1. ∎

Corollary 3.3. *Every θ-closed set is rw-closed but not conversely.*

Proof. Follows from Velicko [35] and Corollary 3.1. ∎

Corollary 3.4. *Every δ-closed set is rw-closed but not conversely.*

Proof. Follows from Velicko [35] and Corollary 3.1. ∎

Corollary 3.5. *Every rw-closed set is gpr-closed but not conversely.*

Proof. Follows from Gnanambal [15] and Theorem 3.2. ∎

Corollary 3.6. *Every π-closed set is rw-closed but not conversely.*

Proof. Follows from Dontchev and Noiri [11] and Corollary 3.1. ∎

Theorem 3.3. *Every rw-closed set is regular weakly generalized closed but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular open set is regular semiopen. ∎

The converse of the above theorem need not be true as seen from the following example.

Example 3.3. Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the set $A = \{b, c\}$ is regular weekly generalized closed, but not rw-closed in X.

Remark 3.1. The following example shows that rw-closed sets are independent of g*-closed sets, mildly g-closed sets, g-closed sets, wg-closed sets, semiclosed sets, α-closed sets, gα-closed sets, αg-closed sets, sg-closed sets, gs-closed sets, gsp-closed sets, β-closed sets, pre-closed sets, gp-closed sets, swg-closed sets, πg-closed sets, θ-generalized closed sets and δ-generalized closed sets.

Example 3.4. Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then

(i) Closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, c, d\}\{b, c, d\}$.

(ii) Rw-closed sets in (X, τ) are $X, \emptyset, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c, \}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(iii) g*-closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(iv) Mildly g-closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}$.

(v) g-closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(vi) Wg-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(vii) Semiclosed sets in (X, τ) are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, c\}, \{a, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}$.

(viii) α-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.

(ix) gα-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}$.

(x) αg-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(xi) Sg-closed sets in (X, τ) are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{c, d\}, \{b, c\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, c, d\}, \{b, c, d\}$.

(xii) Gs-closed sets in (X, τ) are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}$.

(xiii) Gsp-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(xiv) β-closed sets in (X, τ) are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, c\}, \{a, c\}, \{b, d\}, \{b, c, d\}, \{a, c, d\}$.

(xv) Pre-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}$.

(xvi) Gp-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}$.

(xvii) Swg-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.

(xviii) πg-closed sets in (X, τ) are $X, \emptyset, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(xix) θ-generalized closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(xx) δ-generalized closed sets in (X, τ) are $X, \emptyset, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

Remark 3.2. From the above discussion and known results we have the following implications (Figure 1).

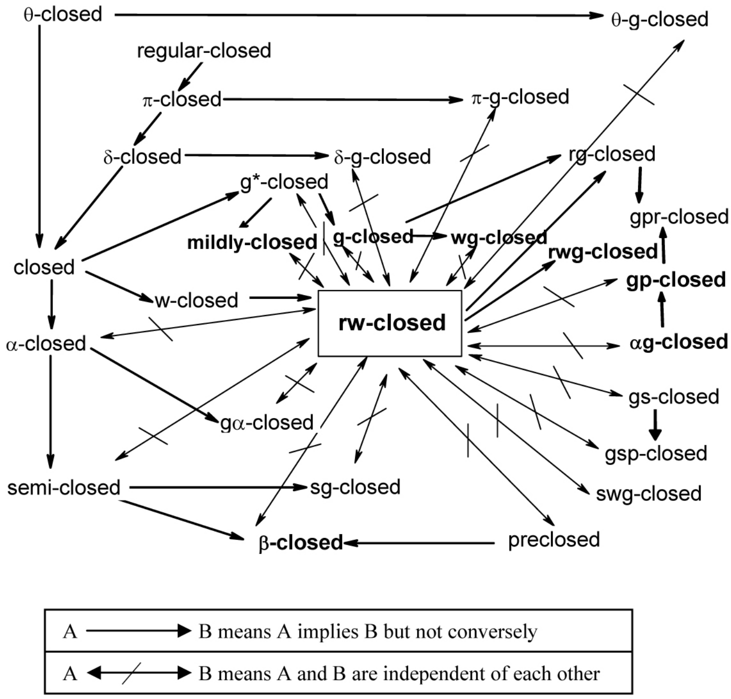

Figure 1

Theorem 3.4. *The union of two rw-closed subsets of X is also an rw-closed subset of X.*

Proof. Assume that A and B are rw-closed sets in X. Let U be regular semiopen in X such that $A \cup B \subset U$. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are rw-closed, $\mathrm{cl}(A) \subset U$ and $\mathrm{cl}(B) \subset U$. Hence $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \subset U$. That is $\mathrm{cl}(A \cup B) \subset U$. Therefore $A \cup B$ is an rw-closed set in X. ∎

Remark 3.3. The intersection of two rw-closed sets in X is generally not an rw-closed set in X.

Example 3.5. Let $X = \{a, b, c, d\}$ be with $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. If $A = \{a, b\}$ and $B = \{a, c, d\}$, then A and B are rw-closed sets in X, but $A \cap B = \{a\}$ is not an rw-closed set in X.

Theorem 3.5. *If a subset A of X is rw-closed in X, then $\mathrm{cl}(A) \setminus A$ does not contain any nonempty regular semiopen set in X.*

Proof. Suppose that A is an rw-closed set in X. We prove the result by contradiction. Let U be a regular semiopen set such that $\mathrm{cl}(A) \setminus A \supset U$ and $U \neq \emptyset$. Now $U \subset \mathrm{cl}(A) \setminus A, U \subset X \setminus A$ which implies $A \subset X \setminus U$. Since U is regular semiopen, by Lemma 2.2, $X \setminus U$ is also regular semiopen in X. Since A is an rw-closed set in X, by definition, we have $\mathrm{cl}(A) \subset X \setminus U$. So $U \subset X \setminus \mathrm{cl}(A)$. Also $U \subset \mathrm{cl}(A)$. Therefore $U \subset (\mathrm{cl}(A) \cap (X \setminus \mathrm{cl}(A))) = \emptyset$. This shows that $U = \emptyset$ which is a contradiction. Hence $\mathrm{cl}(A) \setminus A$ does not contain any nonempty regular semiopen set in X. ∎

The converse of the above theorem need not be true as seen from the following example.

Example 3.6. If $\mathrm{cl}(A) \setminus A$ contains no nonempty regular semiopen subset in X, then A need not be rw-closed. Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{a\}$. Then $\mathrm{cl}(A) \setminus A = \{a, c\} \setminus \{a\} = \{c\}$ does not contain any nonempty regular semiopen set, but A is not an rw-closed set in X.

Corollary 3.7. *If a subset A of X is an rw-closed set in X, then $\mathrm{cl}(A) \setminus A$ does not contain any nonempty regular open set in X, but not conversely.*

Proof. Follows from Theorem 3.5 and the fact that every regular open set is regular semiopen. ∎

Corollary 3.8. *If a subset A of X is an rw-closed set in X, then $\mathrm{cl}(A) \setminus A$ does not contain any nonempty regular closed set in X, but not conversely.*

Proof. Follows from Theorem 3.5 and the fact that every regular open set is regular semiopen. ∎

Theorem 3.6. *For an element $x \in X$, the set $X \setminus \{x\}$ is rw-closed or regular semiopen.*

Proof. Suppose $X \setminus \{x\}$ is not regular semiopen. Then X is the only regular semiopen set containing $X \setminus \{x\}$. This implies $\mathrm{cl}(X \setminus \{x\}) \subset X$. Hence $X \setminus \{x\}$ is an rw-closed set in X. ∎

Theorem 3.7. *If A is regular open and rw-closed, then A is regular closed and hence clopen.*

Proof. Suppose A is regular open and rw-closed. As every regular open set is regular semiopen and $A \subset A$, we have $\mathrm{cl}(A) \subset A$. Also $A \subset \mathrm{cl}(A)$. Therefore $\mathrm{cl}(A) = A$. That means A is closed. Since A is regular open, A is open. Now $\mathrm{cl}(\mathrm{int}(A)) = \mathrm{cl}(A) = A$. Therefore A is regular closed and clopen. ∎

Theorem 3.8. *If A is an rw-closed subset of X such that $A \subset B \subset \mathrm{cl}(A)$, then B is an rw-closed set in X.*

Proof. Let A be an rw-closed set of X such that $A \subset B \subset \mathrm{cl}(A)$. Let U be a regular semiopen set of X such that $B \subset U$. Then $A \subset U$. Since A is rw-closed, we have $\mathrm{cl}(A) \subset U$. Now $\mathrm{cl}(B) \subset \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A) \subset U$. Therefore B is an rw-closed set in X. ∎

Remark 3.4. The converse of Theorem 3.8 need not be true in general. Consider the topological space (X, τ), where $X = \{a, b, c, d\}$ be with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $A = \{d\}$ and $B = \{c, d\}$. Then A and B are rw-closed sets in (X, τ), but $A \subset B$ is not subset in $\mathrm{cl}(A)$.

Theorem 3.9. *Let A be rw-closed in (X, τ). Then A is closed if and only if $\mathrm{cl}(A) \setminus A$ is regular semiopen.*

Proof. Suppose A is closed in X. Then $\mathrm{cl}(A) = A$ and so $\mathrm{cl}(A) \setminus A = \emptyset$, which is regular semiopen in X. ∎

Conversely, suppose $\mathrm{cl}(A) \setminus A$ is regular semiopen in X. Since A is rw-closed, by Theorem 3.8, $\mathrm{cl}(A) \setminus A$ does not contain any nonempty regular semiopen set in X. Then $\mathrm{cl}(A) \setminus A = \emptyset$, hence A is closed in X.

Theorem 3.10. *If A is regular open and rg-closed, then A is rw-closed in X.*

Proof. Let A be regular open and rg-closed in X. We prove that A is an rw-closed set in X. Let U be any regular semiopen set in X such that $A \subset U$. Since A is regular open and rg-closed, we have $\mathrm{cl}(A) \subset A$. Then $\mathrm{cl}(A) \subset A \subset U$. Hence A is rw-closed in X. ∎

Theorem 3.11. *If a subset A of a topological space X is both regular semiopen and rw-closed, then it is closed.*

Proof. Suppose a subset A of a topological space X is both regular semiopen and rw-closed. Now $A \subset A$. Then $\mathrm{cl}(A) \subset A$. Hence A is closed. ∎

Corollary 3.9. *Let A be regular semiopen and rw-closed in X. Suppose that F is closed in X. Then $A \cap F$ is an rw-closed set in X.*

Proof. Let A be regular semiopen and rw-closed in X and F be closed. By Theorem 3.11, A is closed. So $A \cap F$ is closed and hence $A \cap F$ is an rw-closed set in X. ∎

The next two results are required in the sequel.

Theorem 3.12. [7] *If A is open and S is semiopen in a topological space X, then $A \cap S$ is semiopen in X.*

Theorem 3.13. [18] *Let $A \subset Y \subset X$, where X is a topological space and Y is subspace of X. If $A \in SO(X)$, then $A \in SO(Y)$.*

Lemma 3.1. *Let $A \subset Y \subset X$, where X is a topological space and Y is an open subspace of X. If $A \in RSO(X)$, then $A \in RSO(Y)$.*

Proof. Suppose that $A \subset Y \subset X$, Y is an open subspace of X and $A \in RSO(X)$. We prove that A is regular semiopen in Y, i.e. A is both semiopen and semiclosed in Y. Now we show that A is semiopen in Y. Since $A \subset Y \subset X$ and A is semiopen in X, by Theorem 3.13, A is semiopen in Y. Now we show that A is semiclosed in Y. We have $Y \setminus A = Y \cap (X \setminus A)$. Since A is regular semiopen in X, by Lemma 2.2, $X \setminus A$ is also regular semiopen in X and so $X \setminus A$ is semiopen in X. Since Y is open in X, by Theorem 3.12, $Y \cap (X \setminus A)$ is semiopen in X. That is $Y \setminus A$ is semiopen in X. Also $Y \setminus A \subset Y \subset X$. By Theorem 3.13, $Y \setminus A$ is semiopen in Y and so A is semiclosed in Y. Thus A is both semiopen and semiclosed in Y. Hence A is regular semiopen in Y. ∎

Theorem 3.14. *Suppose $B \subset A \subset X$, B is rw-closed set relative to A and A is both regular open and rw-closed subset of X. Then B is rw-closed set relative to X.*

Proof. Let $B \subset G$ and G be a regular semiopen set in X. But it is given that $B \subset A \subset X$, and therefore $B \subset A \cap G$. Now we show that $A \cap G$ is regular semiopen in A. First we show that $A \cap G$ is regular semiopen in X. Since A is open and G is semiopen in X, by Theorem 3.12, $A \cap G$ is semiopen in X. Since A is regular open and rw-closed by Theorem 3.7, A is closed and so A is semiclosed in X. Also

G is semiclosed in X, as every regular semiopen set is semiclosed. Then $A \cap G$ is semiclosed in X. Thus $A \cap G$ is both semiopen and semiclosed in X and hence $A \cap G$ is a regular semiopen in X. Also $A \cap G \subset A \subset X$ and A is open subspace of X, by Lemma 3.1, $A \cap G$ is regular semiopen in A. Since B is rw-closed relative to A, $\mathrm{cl}_A(B) \subset A \cap G$ (i). But $\mathrm{cl}_A(B) = A \cap \mathrm{cl}(B)$ (ii). From (i) and (ii) it follows that $A \cap \mathrm{cl}(B) \subset A \cap G$. Consequently $A \cap \mathrm{cl}(B) \subset G$. Since A is regular open and rw-closed, by Theorem 3.7, $\mathrm{cl}(A) = A$ and so $\mathrm{cl}(B) \subset A$. We have $A \cap \mathrm{cl}(B) = cl(B)$. Thus $\mathrm{cl}(B) \subset G$ and hence B is rw-closed relative to X. ∎

Corollary 3.10. [26] *Let A be regular open in X and B be a subset of A. Then B is regular open in X if and only if B is regular open in the subspace A.*

Lemma 3.2. *Let Y be regular open in X and U be a subset of Y. Then U is regular semiopen in X if and only if U is regular semiopen in the subspace Y.*

Proof. Suppose U is regular semiopen in X. Then $U \subset Y \subset X$ and Y is open subspace of X. By Lemma 3.1, U is regular semiopen in Y. Conversely, suppose U is regular semiopen in the subspace Y. We prove that U is regular semiopen in X. By definition, there exists a regular open set V in Y such that $V \subset U \subset \mathrm{cl}_Y(V)$. That is $V \subset U \subset \mathrm{cl}_Y(V) = Y \cap \mathrm{cl}(V) \subset \mathrm{cl}(V)$. Now $V \subset U \subset \mathrm{cl}(V)$. Also by Corollary 3.10, V is regular open in X. Therefore U is regular semiopen in X. ∎

Theorem 3.15. *Let $A \subset Y \subset X$ and suppose that A is rw-closed in X. Then A is rw-closed in Y provided Y is regular open in X.*

Proof. Let A be rw-closed in X and Y be regular open subspace of X. Let U be any regular semiopen set in Y such that $A \subset U$. Then $A \subset Y \subset X$. By Lemma 3.2, U is regular semiopen in X. Since A is rw-closed in X, $\mathrm{cl}(A) \subset U$. That is $Y \cap \mathrm{cl}(A) \subset Y \cap U = U$. Thus $\mathrm{cl}_Y(A) \subset U$. Hence A is rw-closed in Y. ∎

Theorem 3.16. *If A is both open and g-closed in X, then it is rw-closed in X.*

Proof. Let A be an open and g-closed set in X. Let $A \subset U$ and let U be regular semiopen in X. Now $A \subset A$. By hypothesis $\mathrm{cl}(A) \subset A$. That is $\mathrm{cl}(A) \subset U$. Thus A is rw-closed in X. ∎

Remark 3.5. If A is both open and rw-closed in X, then A need not be g-closed, in general, as seen from the following example.

Example 3.7. Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this topological space the subset $\{a, b\}$ is open and rw-closed, but not g-closed.

Theorem 3.17. *If a subset A of a topological space is both open and wg-closed, then it is rw-closed.*

Proof. Suppose a subset A of X is both open and wg-closed. Let $A \subset U$ with U regular semiopen in X. Now $A \supset \mathrm{cl}(\mathrm{int}(A)) = A$, as A is open. That is $\mathrm{cl}(A) \subset A \subset U$. Thus A is an rw-closed set in X. ∎

Theorem 3.18. *In a topological space X, if $RSO(X) = \{X, \emptyset\}$, then every subset of X is an rw-closed set.*

Proof. Let X be a topological space and $RSO(X) = \{X, \emptyset\}$. Let A be any subset of X. Suppose $A = \emptyset$. Then $\emptyset$ is an rw-closed set in X. Suppose $A \neq \emptyset$. Then X is the only regular semiopen set containing A and so $\mathrm{cl}(A) \subset X$. Hence A is an rw-closed set in X. ∎

Remark 3.6. The converse of Theorem 3.18 need not be true in general as seen from the following example.

Example 3.8. Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{X, \emptyset, \{a, b\}, \{c, d\}\}$.Then every subset of X is an rw-closed set in X, but $RSO(X) = \{X, \emptyset, \{a, b\}, \{c, d\}\}$.

Theorem 3.19. *In a topological space (X, τ), $RSO(X, \tau) \subset \{F \subset X : F^c \in \tau\}$ if and only if every subset of (X, τ) is rw-closed.*

Proof. Suppose that $RSO(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. Let A be any subset of (X, τ) such that $A \subset U$, where U is regular semiopen. Then $U \in RSO(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. That is $U \in \{F \subset X : F^c \in \tau\}$. Thus U is closed. Then $\mathrm{cl}(U) = U$. Also $\mathrm{cl}(A) \subset \mathrm{cl}(U) = U$. Hence A is an rw-closed set in X. Conversely, suppose that every subset of (X, τ) is rw-closed. Let $U \in RSO(X, \tau)$. Since $U \subset U$ and U is rw-closed, we have $\mathrm{cl}(U) \subset U$. Thus $\mathrm{cl}(U) = U$ and $U \in \{F \subset X : F^c \in \tau\}$. Therefore $RSO(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. ∎

Theorem 3.20. *Let X be a regular space in which every regular semiopen subset is open. If A is a compact subset of X, then A is rw-closed.*

Proof. Suppose $A \subset U$ and U is regular semiopen. By hypothesis U is open. But A is a compact subset in the regular space X. Hence there exists an open set V such that $A \subset V \subset \mathrm{cl}(V) \subset U$. Now $A \subset \mathrm{cl}(V)$ implies $\mathrm{cl}(A) \subset \mathrm{cl}(\mathrm{cl}(V)) = \mathrm{cl}(V) \subset U$. That is $\mathrm{cl}(A) \subset U$. Hence A is rw-closed in X. ∎

Definition 3.2. *The intersection of all regular semiopen subsets of (X, τ) containing A is called the regular semi-kernel of A and is denoted by $rsker(A)$.*

Lemma 3.3. *Let X be a topological space and A be a subset of X. If A is regular semiopen in X, then $rsker(A) = A$, but not conversely.*

Proof. Follows from Definition 3.2. ∎

Lemma 3.4. *For any subset A of (X, τ), $sker(A) \subset rsker(A)$.*

Proof. Follows from the implication $RSO(X) \subset SO(X)$. ∎

Lemma 3.5. *For any subset A of (X, τ), $A \subset rsker(A)$.*

Proof. Follows from Definition 3.2. ∎

Theorem 3.21. *A subset A of (X, τ) is rw-closed if and only if* $\mathrm{cl}(A) \subset rsker(A)$.

Proof. Suppose that A is rw-closed. Then $\mathrm{cl}(A) \subset U$, whenever $A \subset U$ and U is regular semiopen. Let $x \in \mathrm{cl}(A)$. Suppose $x \notin rsker(A)$; then there is a regular semiopen set U containing A such that x is not in U. Since A is rw-closed, $\mathrm{cl}(A) \subset U$. We have x not in $\mathrm{cl}(A)$, which is a contradiction. Hence $x \in rsker(A)$ and so $\mathrm{cl}(A) \subset rsker(A)$. Conversely, let $\mathrm{cl}(A) \subset rsker(A)$. If U is any regular semiopen set containing A, then $rsker(A) \subset U$. That is $\mathrm{cl}(A) \subset rsker(A) \subset U$. Therefore A is rw-closed in X. ∎

Lemma 3.6. [17] *Let x be a point of (X, τ). Then $\{x\}$ is either nowhere dense or preopen.*

Remark 3.7. [13] In the notation of Lemma 3.6, we may consider the following decomposition of a given topological space (X, τ), namely $X = X_1 \cup X_2$ where $X_1 = \{x \in X : \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is preopen$\}$.

Lemma 3.7. *For any subset A of (X, τ), $X_2 \cap \mathrm{cl}(A) \subset rsker(A)$.*

Proof. Let $x \in X_2 \cap \mathrm{cl}(A)$ and suppose that x is not in $rsker(A)$. Then there is a regular semiopen set U containing A such that x is not in U. If $F = X \setminus U$, then F is regular semiopen and so F is semiclosed. Now $\mathrm{scl}(\{x\}) = \{x\} \cup \mathrm{int}(\mathrm{cl}(\{x\})) \subset F$. Since $\mathrm{cl}(\{x\}) \subset \mathrm{cl}(A)$, we have $\mathrm{int}(\mathrm{cl}(\{x\}) \subset A \cup \mathrm{int}(\mathrm{cl}(A))$. Again since $x \in X_2$, we have x not in X_1 and so $\mathrm{int}(\mathrm{cl}(\{x\}) \neq \emptyset$. Therefore there has to be some point $y \in A \cup \mathrm{int}(\mathrm{cl}(\{x\}))$ and hence $y \in F \cap A$, which is a contradiction. So $x \in rsker(A)$ and hence $X_2 \cap \mathrm{cl}(A) \subset rsker(A)$. ∎

Theorem 3.22. *For any subset A of (X, τ), if $X_1 \cap \mathrm{cl}(A) \subset A$, then A is rw-closed in X.*

Proof. Let A be any subset of (X, τ). Suppose that $X_1 \cap \mathrm{cl}(A) \subset A$. We prove that A is rw-closed in X. Then $\mathrm{cl}(A) \subset rsker(A)$, since $A \subset rsker(A)$, by Lemma 3.5. Now $\mathrm{cl}(A) = X \cap \mathrm{cl}(A) = (X_1 \cup X_2) \cap \mathrm{cl}(A)$. That is $\mathrm{cl}(A) = (X_1 \cap \mathrm{cl}(A)) \cup (X_2 \cap \mathrm{cl}(A)) \subset rsker(A)$, since $X_1 \cap \mathrm{cl}(A) \subset rsker(A)$ and by Lemma 3.7, $X_2 \cap \mathrm{cl}(A) \subset rsker(A)$. That is $\mathrm{cl}(A) \subset rsker(A)$. By Theorem 3.21, A is rw-closed in X. ∎

The converse of the above theorem need not be true in general as seen from the following example.

Example 3.9. Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Here $X_1 = \{c, d\}$ and $X_2 = \{a, b\}$. Take $A = \{a, b, c\}$. Then A is rw-closed in X. But $X_1 \cap \mathrm{cl}(A) = \{c, d\} \cap X = \{c, d\}$ is not subset in A.

Definition 3.3. *A subset A in (X, τ) is called regular w-open (briefly rw-open) in X if A^c is rw-closed in (X, τ).*

Theorem 3.23. *Every singleton point set in a space is either rw-open or regular semiopen.*

Proof. Let X be a topological space. Let $x \in X$. We prove $\{x\}$ is either rw-open or regular semiopen, i.e. $X \setminus \{x\}$ is either rw-closed or regular semiopen, which follows from Theorem 3.6. ∎

References

[1] M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.* **12**(1983), 77–90.

[2] D. Andrijevic, Semi-preopen sets, *Mat. Vesnik* **38**(1986), 24–32.

[3] S.P. Arya and T.M. Nour, Characterizations of s-normal spaces, *Indian J. Pure Appl. Math.* **21**(1990), 717–719.

[4] P. Bhattacharyya and B.K. Lahiri, Semi-generalized closed sets in topology, *Indian J. Math.* **29**(1987), 376–382.

[5] N. Biswas, On characterization of semi-continuous functions, *Atti Accad. Naz. Lincei Rend, Cl. Sci. Fis. Mat. Natur.* **48**(8)(1970), 399–402.

[6] D.E. Cameron, Properties of S-closed spaces, *Proc. Amer Math. Soc.* **72**(1978), 581–586.
[7] S.G. Crossley and S.K. Hildebrand, Semi-closure, *Texas J. Sci.* 22(1971), 99–112.
[8] G. Di Maio and T. Noiri, On s-closed spaces, *Indian J. Pure Appl. Math.* **18**(3)(1987), 226–233.
[9] J. Dontchev, On generalizing semi-preopen sets, *Mem. Fac Sci. Kochi. Univ. Ser. A. Math.* **16**(1995), 35–48.
[10] J. Dontchev and M. Ganster, On δ-generalized set $T_{\frac{3}{4}}$ spaces, *Mem. Fac. Sci. Kochi Uni. Ser. A. Math.* **17**(1996), 15–31.
[11] J. Dontchev and T. Noiri, Quasi-normal spaces and πg-closed sets, *Acta Math. Hungar.* **89**(3)(2000), 211–219.
[12] J. Dontchev and H. Maki, On θ-generalized closed sets, *Topology Atlass*, www.Unipissing.ca/topology/p/a/b/a/08.htm.
[13] J. Dontchev and H. Maki, On sg-closed sets and semi-λ closed sets, *Questions and Answers Gen. Topology* **15**(1997), 253–266.
[14] G.L. Garg and D. Sivaraj, On sc-compact and S-closed spaces, *Boll. Un. Mat. Ital.* **6(**3B)(1984), 321–332.
[15] Y. Gnanambal, On generalized preregular closed sets in topological spaces, *Indian J. Pure App. Math.* **28**(1997), 351–360.
[16] Y. Gnanambal and K. Balachandran, On gpr-continuous functions in topological spaces, *Indian J. Pure Appl. Math.* **30**(6)(1999), 581–593.
[17] D.S. Jankovic and I.L. Reilly, On semi separation properties, *Indian J. Pure Appl. Math.* **16**(1985), 957–964.
[18] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70**(1963), 36–41.
[19] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* **19**(1970), 89–96.
[20] H. Maki, J. Umehara and T. Noiri, Every topological space is pre-$T_{\frac{1}{2}}$, *Mem. Fac. Sci. Kochi Univ. Ser. A. Math.* **17**(1996), 33–42.
[21] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, *Mem. Sci. Kochi Univ. Ser. A. Math.* **15**(1994), 51–63.
[22] H. Maki, R. Devi and K. Balachandran, Generalized α-closed sets in topology, *Bull. Fukuoka Univ. Ed. part-III* **42**(1993), 13–21.
[23] A.S. Mashhour, I.A. Hasanein and S.N. El-Deeb, α-open mappings, *Acta. Math. Hungar.* **41**(1983), 213–218.
[24] A.S. Mashhour, M.E. Abd. El-Monsef and S.N. El-Deeb, On pre continuous mappings and weak pre-continuous mappings, *Proc Math, Phys. Soc. Egypt* **53**(1982), 47–53.
[25] N. Nagaveni, *Studies on Generalizations of Homeomorphisms in Topological Spaces*, Ph.D. Thesis, Bharathiar University, Coimbatore, 1999.
[26] T.Noiri, On almost open mappings, *Memoir Miyakonojo Tech. College* **7**(1972).
[27] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.* **15**(1965), 961–970.
[28] N. Palaniappan and K.C. Rao, Regular generalized closed sets, *Kyungpook Math. J.* **33**(1993), 211–219.
[29] J.K. Park and J.H. Park, Mildly generalized closed sets, almost normal and mildly normal spaces, *Chaos, Solitions and Fractals* **20**(2004), 1103–1111.
[30] A. Pushpalatha, *Studies on Generalizations of Mappings in Topological Spaces*, Ph.D. Thesis, Bharathiar University, Coimbatore, 2000.
[31] P. Sundaram and M. Sheik John, On w-closed sets in topology, *Acta Ciencia Indica* **4**(2000), 389–392.
[32] M. Stone, Application of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* **41**(1937), 374–481.
[33] J. Tong, Weak almost continuous mapping and weak nearly compact spaces, *Boll. Un. Mat. Ital.* **6**(1982), 385–391.
[34] M.K.R.S. Veera Kumar, Between closed sets and g-closed sets, *Mem. Fac. Sci. Kochi Univ. (Math.)* **21**(2000), 1–19.
[35] N.V. Velicko, H-closed topological spaces, *Trans. Amer. Math. Soc.* **78**(1968), 103–118.