Linear Differential Polynomials Sharing Three Values with Finite Weight

ABHIJIT BANERJEE
Department of Mathematics, Kalyani Government Engineering College,
West Bengal 741235, India
abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com

Abstract. In the paper we study the uniqueness problem of two linear differential polynomials with weighted sharing of three values which improve and supplement a recent result of Lahiri-Banerjee [10].

2000 Mathematics Subject Classification: 30D35

Key words and phrases: Uniqueness, Weighted sharing, Linear differential polynomial.

1. Introduction and definitions

Let $f$ and $g$ be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f$ and $g$ have same set of $a$-points with the same multiplicities, we say that $f$ and $g$ share the value $a$ CM (counting multiplicities) and if we do not consider the multiplicities then $f, g$ are said to share the value $a$ IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [3].

**Definition 1.1.** [10] We denote by $N(r, a; f) = 1$ the counting function of simple $a$-points of $f$ for $a \in \mathbb{C} \cup \{\infty\}$.

**Definition 1.2.** [10] Let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r, a; f \geq p)$ the counting function of those distinct $a$-points of $f$ whose multiplicities are not less than $p$.

Let $a_1, a_2, \ldots, a_n (a_n \neq 0)$ be finite complex numbers. In this paper we shall denote by $F$ and $G$ the following two linear differential polynomials unless otherwise stated.

$$F = \sum_{i=1}^{n} a_i f^{(i)} \quad \text{and} \quad G = \sum_{i=1}^{n} a_i g^{(i)}$$

In [4] the following result is proved.

Received: July 4, 2006; Revised: December 13, 2006.
Theorem 1.1. [4] Let $f$ and $g$ be two nonconstant meromorphic functions. If

(i) $f$ and $g$ share $\infty$ CM;
(ii) $F$ and $G$ share $0, 1$ CM;
(iii) $\sum_{a \neq \infty} \delta(a; f) \left(1 + n(1 - \Theta(\infty; f)) \right) - \frac{3(1 - \Theta(\infty; f))}{2} \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2},$

where $\sum_{a \neq \infty} \delta(a; f) > 0,$

then either (a) $F \equiv G$ or (b) $F \cdot G \equiv 1.$ If further, $f$ has at least one pole or $F$ has at least one zero, the case (b) does not arise.

In [1] Fang and Lahiri improved Theorem 1.1. To state their result we require the following definition.

Definition 1.3. [5, 6] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $1 + k$ times if $m > k.$ If $E_k(a; f) = E_k(a; g),$ we say that $f, g$ share the value $a$ with weight $k.$

The definition implies that if $f, g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ if and only if it is an $a$-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an $a$-point of $f$ if and only if it is an $a$-point of $g$ with multiplicity $m(> k),$ where $m$ is not necessarily equal to $n.$

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k.$ Clearly if $f, g$ share $(a, k)$ then $f, g$ share $(a, p)$ for all integer $p, 0 \leq p < k.$ Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

With the notion of weighted sharing of values improving Theorem 1.1 the following result was proved in [1].

Theorem 1.2. [1] Let $f$ and $g$ be two nonconstant meromorphic functions. If

(i) $f$ and $g$ share $(\infty, \infty),$
(ii) $F$ and $G$ share $(0, 1), (1, \infty),$
(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2},$

then either (a) $F \equiv G$ or (b) $F \cdot G \equiv 1.$ If, further, $f$ has at least one pole or $F$ has at least one zero, the case (b) does not arise.

Recently Lahiri and Banerjee [10] reduced the weight of sharing values in Theorem 1.2 and proved the following two theorems.

Theorem 1.3. Let $f$ and $g$ be two nonconstant meromorphic functions. If

(i) $f$ and $g$ share $(\infty, 1),$
(ii) $F$ and $G$ share $(0, 1), (1, 6),$
(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$

then either (a) $F \equiv G$ or (b) $F \cdot G \equiv 1.$ If, further, $f$ has at least one pole or $F$ has at least one zero, the case (b) does not arise.
Theorem 1.4. Let $f$ and $g$ be two nonconstant meromorphic functions. If

(i) $f$ and $g$ share $(\infty, 0)$,

(ii) $F$ and $G$ share $(0, 1), (1, 6)$, where $F = \sum_{i=1}^{n} a_i f^{(i)}$, $G = \sum_{i=1}^{n} a_i g^{(i)}$ and $n \geq 2$,

(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$

then either (a) $F \equiv G$ or (b) $F \cdot G \equiv 1$. If further, $f$ has at least one pole or $F$ has at least one zero, the case (b) does not arise.

In the present paper we investigate the situation of further reducing the weight of the value 1 in the above two theorems.

We now give some more definitions.

Definition 1.4. [2] For a meromorphic function $f$ we put

$$T_0(r, f) = \int_1^r \frac{T(t, f)}{t} dt, \quad N_0(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt,$$

$$m_0(r, a; f) = \int_1^r \frac{m(t, a; f)}{t} dt, \quad S_0(r, f) = \int_1^r \frac{S(t, f)}{t} dt,$$

etc. where $a \in \mathbb{C} \cup \{\infty\}$

Definition 1.5. [2] For a meromorphic function $f$ we put

$$\delta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a; f)}{T_0(r, f)} = \liminf_{r \to \infty} \frac{m_0(r, a; f)}{T_0(r, f)}.$$

Definition 1.6. [6, 9] Let $f, g$ share a value a IM. We denote by $N_*(r, a; f, g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$. Clearly $N_*(r, a; f, g) \equiv N_*(r, a; g, f)$.

Definition 1.7. [11] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \neq g = b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are $b$-points of $g$.

Definition 1.8. [11] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \neq g \neq b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are not the $b$-points of $g$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Henceforth we shall denote by $H, \Phi_1, \Phi_2, \Phi_3$ the following four functions.

$$H = \frac{F''}{F'} - \frac{2F'}{F - 1} - \frac{G''}{G'} + \frac{2G'}{G - 1},$$

$$\Phi_1 = \frac{F'}{F(F - 1)} - \frac{G'}{G(G - 1)} = \left(\frac{F'}{F - 1} - \frac{G'}{G - 1}\right) - \left(\frac{F'}{F} - \frac{G'}{G}\right),$$

$$\lim_{r \to \infty} \frac{S_0(r, f)}{T_0(r, f)} = 0$$

through all values of $r$.

Lemma 2.2. [8, 9] If $F, G$ share $(0, 0), (1, 0), (\infty, 0)$ then

(i) $T(r, F) \leq 3T(r, G) + S(r, F),$

(ii) $T(r, G) \leq 3T(r, F) + S(r, G).$

Lemma 2.2 shows that $S(r, F) = S(r, G)$ and we denote them by $S(r)$.

Lemma 2.3. [13] Let $F, G$ share $(0, 0), (1, 0), (\infty, 0)$ and $H \equiv 0$ then $F, G$ share $(0, \infty), (1, \infty), (\infty, \infty)$.

Lemma 2.4. [7] Let $F, G$ share $(1, 1)$ and $H \not\equiv 0$ then

$$N(r, 1; F) = 1 = N(r, 1; G) = 1 \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.5. [14] Let $f, g$ share $(0, k_1), (\infty, k_2)$ and $(1, k_3)$ where $k_j (j = 1, 2, 3)$ are positive integers satisfying $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$. Then

$$\overline{N}(r, 0; |f| \geq 2) + \overline{N}(r, \infty; |f| \geq 2) + \overline{N}(r, 1; |f| \geq 2) = S(r).$$

Lemma 2.6. Let $F, G$ share $(0, 1), (1, 2), (\infty, 1)$. If $F \not\equiv G$ and $\overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r)$ then for $a = 0, 1$ we get $\overline{N}(r, a; |F| \geq 2) = \overline{N}(r, a; |G| \geq 2) = S(r)$

Proof. We prove $\overline{N}(r, a; |F| \geq 2) = S(r)$ for $a = 0, 1$ because the other can similarly be proved. We suppose that $\overline{N}(r, a; |F| \not\equiv S(r)$ for $a = 0, 1$ because otherwise the case is trivial. Since $F \not\equiv G$, it follows that $\Phi_i \not\equiv 0$ for $i = 2, 3$. Now

\[ (2.1) \quad \overline{N}(r, 0; |F| \geq 2) \leq N(r, 0; \Phi_2) \leq T(r, \Phi_2) + O(1) = N(r, \infty; \Phi_2) + S(r) \leq \overline{N}(r, 1; |F| \geq 3) + S(r). \]

Again

\[ (2.2) \quad 2 \overline{N}(r, 1; |F| \geq 3) \leq \overline{N}(r, 1; |F| \geq 3) + \overline{N}(r, 1; |F| \geq 2) \leq N(r, 0; \Phi_3) \leq N(r, \infty; \Phi_3) + S(r) \leq \overline{N}(r, 0; |F| \geq 2) + \overline{N}(r, \infty; |F| \geq 2) + S(r) = \overline{N}(r, 0; |F| \geq 2) + S(r). \]

From (2.1) and (2.2) we get $\overline{N}(r, 0; |F| \geq 2) = S(r)$ and hence from (2.2) we get $\overline{N}(r, 1; |F| \geq 2) = S(r)$. This proves the lemma.
Lemma 2.7. Let $F, G$ share $(0, 1), (1, 1), (∞, 2)$. If $F \neq G$ and $N(r, 1; F) \geq 2 = N(r, 1; G) \geq 2) = S(r)$ then for $a = 0, ∞$ we get $N(r, a; F) \geq 2) = N(r, a; G) \geq 2) = S(r)$.

Proof. We prove $N(r, a; F) \geq 2) = S(r)$ for $a = 0, ∞$ because the other can similarly be proved. We suppose that $N(r, a; F) \neq S(r)$ for $a = 0, ∞$ because otherwise the case is trivial. Since $F \neq G$, it follows that $Φ_i \neq 0$ for $i = 1, 2$. Now

$$2 N(r, ∞; F) \geq 2) \leq N(r, ∞; F) \geq 3) + N(r, ∞; F) \geq 2)$$

$$\leq N(r, 0; F) \geq 2) + N(r, 1; F) \geq 2) + S(r)$$

$$= N(r, 0; F) \geq 2) + S(r).$$

Again

$$N(r, 0; F) \geq 2) \leq N(r, 0; F)$$

$$= N(r, ∞; F) + S(r)$$

$$\leq N(r, ∞; F) \geq 3) + N(r, 1; F) \geq 2) + S(r)$$

$$= N(r, ∞; F) \geq 3) + S(r).$$

From (2.3) and (2.4) we get $N(r, ∞; F) \geq 3) = S(r)$.

So from (2.4) we get $N(r, 0; F) \geq 2) = S(r)$ and hence using (2.3) we have $N(r, ∞; F) \geq 2) = S(r)$. This proves the lemma.

Lemma 2.8. Let $F, G$ share $(0, 1), (1, m), (∞, k)$ and $N(r, 0; F) \geq 2) = N(r, 0; G) \geq 2) = S(r)$. If $F \neq G$ and $mk - 1 > 0$ then for $a = 1, ∞$ we get $N(r, a; F) \geq 2) = N(r, a; G) \geq 2) = S(r)$.

Proof. We prove $N(r, a; F) \geq 2) = S(r)$ for $a = 1, ∞$ because the other can similarly be proved. We suppose that $N(r, a; F) \neq S(r)$ for $a = 1, ∞$ because otherwise the case is trivial. Since $F \neq G$, it follows that $Φ_i \neq 0$ for $i = 1, 3$. Now

$$m N(r, 1; F) \geq m + 1) \leq (m - 1) N(r, 1; F) \geq 1 + m) + N(r, 1; F) \geq 2)$$

$$\leq N(r, 0; F)$$

$$\leq N(r, ∞; F) + S(r)$$

$$\leq N(r, ∞; F) \geq k + 1) + N(r, 0; F) \geq 2) + S(r)$$

$$= N(r, ∞; F) \geq k + 1) + S(r).$$

Also

$$k N(r, ∞; F) \geq k + 1) \leq (k - 1) N(r, ∞; F) \geq k + 1) + N(r, ∞; F) \geq 2)$$

$$\leq N(r, 0; F)$$

$$\leq N(r, ∞; F) + S(r)$$

$$\leq N(r, 1; F) \geq m + 1) + N(r, 0; F) \geq 2) + S(r)$$

$$= N(r, 1; F) \geq m + 1) + S(r).$$
From (2.5) and (2.6) we get \((m - \frac{1}{k}) \geq N(r, 1; F) \geq m + 1 \leq S(r)\)
\[\text{i.e. } N(r, 1; F) = m + 1 = S(r).\] So from (2.6) we obtain \(\overline{N}(r, \infty; F) \geq 2 = S(r).\) Again from (2.5) we get \(\overline{N}(r, 1; F) \geq 2 = S(r).\) This completes the proof of the lemma.

**Lemma 2.9.** (12) If \(N(r, 0; f^{(k)})|f \neq 0\) denotes the counting function of those zeros of \(f^{(k)}\) which are not the zeros of \(f,\) where a zero of \(f^{(k)}\) is counted according to its multiplicity then
\[N(r, 0; f^{(k)})|f \neq 0| \leq k \overline{N}(r, \infty; f) + N(r, 0; f^{|f < k|} + k \overline{N}(r, 0; f^{|f \geq k|} + S(r, f)).\]

**Lemma 2.10.** For a meromorphic function \(F\)
\[\overline{N}(r, 1; F) \geq k + 1 \leq \frac{1}{k} \overline{N}(r, 0; F) + \frac{1}{k} \overline{N}(r, \infty; F) - \frac{1}{k} N_{\otimes}(r, 0; F') + S(r, F),\]
where \(N_{\otimes}(r, 0; F')\) is the counting functions of those zeros of \(F'\) which are not the zeros of \(F(F - 1).\)

**Proof.** Using Lemma 2.9 we get
\[
\begin{align*}
\overline{N}(r, 1; F) & \geq k + 1 \\
& \leq \frac{1}{k} N(r, 0; F^{|F = 1}) \\
& \leq \frac{1}{k} N(r, 0; F^{|F \neq 0}) - \frac{1}{k} N_{\otimes}(r, 0; F') \\
& \leq \frac{1}{k} \overline{N}(r, 0; F) + \frac{1}{k} \overline{N}(r, \infty; F) - \frac{1}{k} N_{\otimes}(r, 0; F') + S(r, F).
\end{align*}
\]
This proves the lemma.

**Lemma 2.11.** Let \(f, g\) share \((\infty; 1), F, G\) share \((0, 1), (1, 3)\) where \(F = \sum_{i=1}^{n} a_i f^{(i)}, G = \sum_{i=1}^{n} a_i g^{(i)}\) and \(n \geq 2.\) If \(F \neq G, N(r, 0; F^{|F = 1}) = N(r, 0; G^{|G = 1}) = S(r)\) and \(N(r, \infty; F^{|F \geq 4}) = N(r, \infty; G^{|G \geq 4}) = S(r)\) then for \(a = 0, 1, \infty\) we get \(N(r, a; F^{|F \geq 2}) = N(r, a; G^{|G \geq 2}) = S(r).\)

**Proof.** We prove \(N(r, a; F^{|F \geq 2}) = S(r)\) for \(a = 0, 1, \infty\) because the other can similarly be proved. We suppose that \(N(r, a; F^{|F}) \neq S(r)\) for \(a = 0, 1, \infty\) because otherwise the case is trivial. Since \(F \neq G,\) it follows that \(\Phi_i \neq 0\) for \(i = 1, 2, 3.\) Since \(f, g\) share \((\infty; 1)\) it follows that \(F, G\) share \((\infty, 3)\) and \(F, G\) has no simple or double pole. i.e. \(N(r, \infty; F) = \overline{N}(r, \infty; G) = \overline{N}(r, \infty; F^{|F \geq 3}) = \overline{N}(r, \infty; G^{|G \geq 3}).\)

So from Lemma 2.10 we get
\[
\begin{align*}
3 \overline{N}(r, \infty; F^{|F \geq 4}) & \leq N(r, 0; \Phi_1) \\
& \leq N(r, \infty; \Phi_1) + S(r) \\
& \leq \overline{N}(r, 1; F^{|F \geq 4}) + \overline{N}(r, 0; F^{|F \geq 2}) + S(r) \\
& \leq \frac{1}{3} N(r, 0; F^{|F = 1}) + \frac{4}{3} \overline{N}(r, 0; F^{|F \geq 2})
\end{align*}
\]
\[ + \frac{1}{3} \overline{N}(r, \infty; F) = 3 \] + \frac{1}{3} \overline{N}(r, \infty; F) \geq 4 \] + S(r), \]

i.e.

\[ \frac{5}{3} \overline{N}(r, \infty; F) = 3 \] \leq \frac{4}{3} \overline{N}(r, 0; F) \geq 2 \] + S(r). \]

Again using Lemma 2.10 we get

\[ \overline{N}(r, 0; F) \geq 2 \leq \overline{N}(r, 0; \Phi_2) \leq \overline{N}(r, \infty; \Phi_2) + S(r) \leq \overline{N}(r, 1; F) \geq 4 + \overline{N}(r, \infty; F) \geq 4 + S(r) \leq \frac{1}{3} \overline{N}(r, 0; F) = 1 + \frac{1}{3} \overline{N}(r, 0; F) \geq 2 \]

\[ + \frac{1}{3} \overline{N}(r, \infty; F) = 3 + S(r) \leq \frac{1}{3} \overline{N}(r, 0; F) \geq 2 \] + \frac{1}{3} \overline{N}(r, \infty; F) = 3 + S(r), \]

i.e.

\[ \overline{N}(r, 0; F) \geq 2 \leq \overline{N}(r, \infty; F) = 3 \] + S(r). \]

Using (2.8) in (2.7) we get

\[ \frac{5}{3} \overline{N}(r, \infty; F) = 3 \leq \frac{2}{3} \overline{N}(r, \infty; F) = 3 \] + S(r)

i.e. \( \overline{N}(r, \infty; F) = 3 = S(r) \), which implies \( \overline{N}(r, \infty; F) \geq 2 = S(r) \). Since \( \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = \overline{N}(r, \infty; F) \geq 3 \) the remaining part of the lemma follows from Lemma 2.6. This completes the proof of the lemma. \( \blacksquare \)

**Lemma 2.12.** Let \( F, G \) share \( (0, 1), (1, 1), (\infty, 1) \) and \( \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r) \) and \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \) then \( \delta_0(0; F) > \frac{1}{2} \).

**Proof.** The lemma can be proved in the similar manner as followed in p. 34 [10]. \( \blacksquare \)

**Lemma 2.13.** [9] Let \( F, G \) share \( (0, 0), (1, 0), (\infty, 0) \) and \( H \neq 0 \). Then

\[ \overline{N}(r, H) \leq \overline{N}_*(r, 0; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) \]

\[ + \overline{N}_*(r, 0; F') + \overline{N}_*(r, 0; G') \]

where \( \overline{N}_*(r, 0; F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F - 1) \) and \( \overline{N}_*(r, 0; G') \) is similarly defined.

**Lemma 2.14.** Let \( F, G \) share \( (0, 1), (1, 1), (\infty, 1) \) and \( H \neq 0 \). If \( \overline{N}(r, a; F) \geq 2 = \overline{N}(r, a; G) \geq 2 = S(r) \) for \( a = 0, 1, \infty \) then \( \delta_0(0; F) \leq \frac{1}{2} \).

**Proof.** Since \( F, G \) have only multiple poles, by the second fundamental theorem we get

\[ T(r, F) + T(r, G) \leq \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + \overline{N}(r, \infty; F) \]

\[ + \overline{N}(r, 0; G) + \overline{N}(r, 1; G) + \overline{N}(r, \infty; G) \]
Let

\[ \sum_{\delta \geq 1} \geq \frac{1}{2} \]

Then either (a) \( F \equiv G \) or (b) \( F G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Theorem 3.2.** Let \( f \) and \( g \) be two nonconstant meromorphic functions. If

(i) \( f \) and \( g \) share \((\infty, 0)\),

(ii) \( F \) and \( G \) share \((0, 2), (1, 3)\),

(iii) \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \),

then either (a) \( F \equiv G \) or (b) \( F G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Theorem 3.3.** Let \( f \) and \( g \) be two nonconstant meromorphic functions. If

(i) \( f \) and \( g \) share \((\infty, 0)\),

(ii) \( F \) and \( G \) share \((0, 1), (1, 2)\),

(iii) \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \),
(iv) \( \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r), \)
then either (a) \( F \equiv G \) or (b) \( F.G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Theorem 3.4.** Let \( f \) and \( g \) be two nonconstant meromorphic functions. If
(i) \( f \) and \( g \) share \((\infty, 1)\),
(ii) \( F \) and \( G \) share \((0, 1), (1, 1)\),
(iii) \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \),
(iv) \( \overline{N}(r, 1; F \geq 2) = \overline{N}(r, 1; G \geq 2) = S(r), \)
then either (a) \( F \equiv G \) or (b) \( F.G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Theorem 3.5.** Let \( f \) and \( g \) be two nonconstant meromorphic functions. If
(i) \( f \) and \( g \) share \((\infty, k-1)\),
(ii) \( F \) and \( G \) share \((0, 1), (1, m)\), where \( mk - 1 > 0 \),
(iii) \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \),
(iv) \( \overline{N}(r, 0; F \geq 2) = \overline{N}(r, 0; G \geq 2) = S(r), \)
then either (a) \( F \equiv G \) or (b) \( F.G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Theorem 3.6.** Let \( f \) and \( g \) be two nonconstant meromorphic functions. If
(i) \( f \) and \( g \) share \((\infty, 1)\),
(ii) \( F \) and \( G \) share \((0, 1), (1, 3)\), where \( F = \sum_{i=1}^{n} a_i f^{(i)} \), \( G = \sum_{i=1}^{n} a_i g^{(i)} \) and \( n \geq 2 \),
(iii) \( \sum_{a \neq \infty} \delta(a; f) > \frac{1}{2} \),
(iv) \( N(r, 0; F = 1) = N(r, 0; G = 1) = S(r) \) and \( \overline{N}(r, \infty; F \geq 4) = \overline{N}(r, \infty; G \geq 4) = S(r), \)
then either (a) \( F \equiv G \) or (b) \( F.G \equiv 1 \). If, further, \( f \) has at least one pole or \( F \) has at least one zero, the case (b) does not arise.

**Example 3.1.** Let \( f = \frac{1}{2}e^{z}(e^{2} - 1) \), \( g = \frac{1}{2}e^{-z} \left( \frac{1}{2} - \frac{1}{6}e^{-z} \right) \) Then \( F = f'' - 3f' = e^{z}(1 - e^{z}) G = g'' - 3g' = e^{-z}(1 - e^{-z}). \) Clearly \( F, G \) share \((0, \infty), (1, \infty)\) and \( f, g \) share \((\infty, \infty)\). Also \( \delta(0; f) = \sum_{a \neq \infty} \delta(a; f) = \frac{1}{2} \) and \( \overline{N}(r, \infty; F) = \overline{N}(r, \infty; F \geq 2) = \overline{N}(r, 1; F \geq 2) = 0. \) Hence we see that the condition (iii) in Theorems 3.1–3.5 is sharp.

**4. Proofs of the theorems**

*Proof of Theorem 3.1.* Since \( f, g \) share \((\infty, 1)\), clearly \( F, G \) share \((\infty, 2)\). Suppose \( H \neq 0 \) then \( F \neq G \). So by Lemma 2.5 and Lemma 2.14 \( \delta_{0}(0; F) \leq \frac{1}{2} \). But by Lemma 2.12 this leads to a contradiction. So \( H \equiv 0 \). Hence by Lemma 2.3 \( F, G \) share \((0, \infty), (1, \infty), (\infty, \infty)\). Now the theorem follows from Theorem 1.2. This proves the theorem.
Proof of Theorem 3.2. Noting that \( F \) and \( G \) have no simple and double pole the theorem can be proved in the line of the proof of Theorem 3.1. This proves the theorem.

Proof of Theorem 3.3. Since \( f, g \) share \((\infty, 0)\), it follows that \( F, G \) share \((\infty, 1)\). Suppose \( H \neq 0 \) then \( F \neq G \). So by Lemma 2.6 and Lemma 2.14 \( \delta_0(0; F) \leq \frac{1}{2} \). But by Lemma 2.12 we have a contradiction. So \( H \equiv 0 \). Hence the theorem follows from Lemma 2.3 and Theorem 1.2. This proves the theorem.

Proof of Theorem 3.4. Clearly \( F, G \) share \((\infty, 2)\). Suppose \( H \neq 0 \) then \( F \neq G \). Using Lemma 2.7 and Lemma 2.14 we have \( \delta_0(0; F) \leq \frac{1}{2} \). Now the theorem can be proved in the line of the proof of Theorem 3.3. This proves the theorem.

Proof of Theorem 3.5. It is clear from the given condition of the theorem \( F, G \) share \((\infty, k)\). Suppose \( H \neq 0 \) then \( F \neq G \). Using Lemma 2.8 and Lemma 2.14 we get \( \delta_0(0; F) \leq \frac{1}{2} \). Now proceeding in the same way as done in Theorem 3.3 we can prove the theorem. This completes the proof of the theorem.

Proof of Theorem 3.6. According to the hypothesis \( F, G \) share \((\infty, 3)\). Suppose \( H \neq 0 \) then \( F \neq G \). Using Lemma 2.11 and Lemma 2.14 we get \( \delta_0(0; F) \leq \frac{1}{2} \). Now proceeding in the same manner as done in Theorem 3.3 we can prove the theorem. This completes the proof of the theorem.

References